2024 Online Physics Olympiad: Open Contest Solutions

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.8 \text{ m/s}^2$ in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI Nspire will not be needed, but they may be used.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before August 25, 2024**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11} \text{ (N} \cdot \text{m}^2\text{)/kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30} \text{ kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1 \text{ u} = 1.66 \cdot 10^{-27} \text{ kg} = 931 \text{ MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34} \text{ J} \cdot \text{s} = 4.41 \cdot 10^{-15} \text{ eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9 \text{ (N} \cdot \text{m}^2\text{)/C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7} \text{ T} \cdot \text{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \text{ (T} \cdot \text{m)/A}$$
- 1 atmospheric pressure,
$$1 \text{ atm} = 1.01 \cdot 10^5 \text{ N/m}^2 = 1.01 \cdot 10^5 \text{ Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8} \text{ W/m}^2\text{/K}^4$$

Problems

1. Jaywalking You are a jaywalker at a distance $x = 8.00$ m from an intersection. At time $t = 0$ you start to cross it at a constant velocity perpendicular to the direction of the street such that it would take you time T to cross it. Incidentally, at $t = 0$, there is also a car at $X = 40.0$ m away from the intersection on the same side as you. It is a self-driving Tesla, and it drives toward the intersection at such a speed that if the light remained green, the car would reach the intersection at $t = T$.

At time $t = 0$, the light turns red, and since the car is driven by a computer, it immediately begins decelerating with constant acceleration a such that it comes to a stop at exactly $X = 0$.

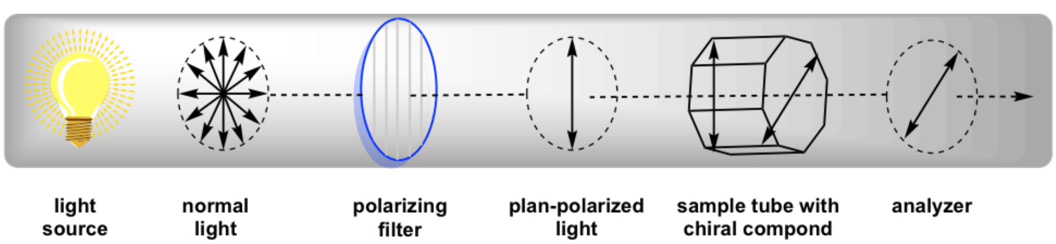
Do you get hit by the car? If yes, report the speed of the car relative to the ground when it hits you, as a percent of its original speed before the light turned red. If no, report how far away the car is from you when you finish crossing the street.

Solution 1: The car's initial velocity is $v_0 = X/T$. Using $v^2 = 2a\Delta x$, we find its acceleration is $a = \frac{X}{2T^2}$. Therefore, the distance the car travels in time T is

$$v_0 T - \frac{1}{2}aT^2 = \frac{3}{4}X = 30.0\ m$$

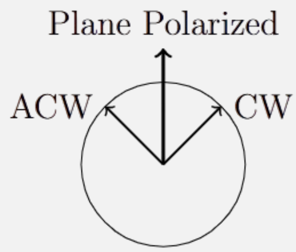
You do not get hit by the car, and you are $\boxed{2.0\text{ m}}$ away from the car when you finish crossing the street.

2. Ambidexterity When plane-polarized light $\lambda = 500$ nm is passed through a solution with chiral molecules (eg. glucose/DNA), the exiting light is observed to have been rotated by an angle $\Delta\theta$. In chemistry, this optical rotation is measured with a polarimeter device. It is used to measure the relative abundances of left-handed and right-handed molecules, each having their own refractive index $n_L = 1.333333$ and $n_R = 1.333338$ affecting left and right circularly polarized respectively.



If the length of the solution container is $L = 0.15$ m, by which angle will the polarization of light rotate, i.e. what is the **total** optical rotation $\Delta\theta$, in radians? Input the smallest positive answer to this question.

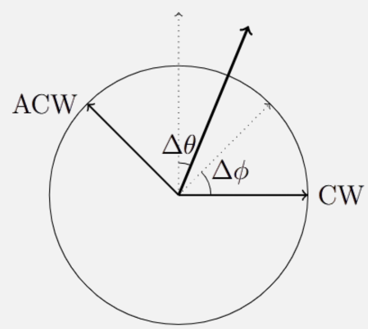
Solution 2:
The key insight is to write the plane polarized light as the superposition of 2 circularly polarized light waves of the same frequency.



After passing through the solution, since they have different refractive indices n_L and n_R, the phase difference between them is given by

$$\frac{\phi}{2\pi} = \frac{L}{\lambda_{\text{medium}}} = \frac{L}{\frac{\lambda}{n}} \implies \Delta\phi = \frac{2\pi}{\lambda} L\Delta n$$

However, note that this is the phase shift and not the rotation angle.



Hence $\Delta\theta = \frac{1}{2}\Delta\phi = \frac{\pi}{\lambda} L\Delta n$. Plugging the values in will give $\boxed{\Delta\theta = \frac{3\pi}{2} \approx 4.71 \text{ rad}}$.

3. Ball Drop A ball with uniform density ρ_b is placed on the surface of a pool with depth d and liquid density $\rho_p < \rho_b$. Another identical ball is lifted a height h above the pool, and then both balls are released at the same time. In order for both balls to touch the bottom of the pool at the same time, the condition $d = nh$ must be met for some dimensionless n that depends on the values of ρ_p and ρ_b. If we define

$$r = \frac{\rho_b - \rho_p}{\rho_b}$$

Then we can express n as

$$n = \frac{Ar^3 + Br^2 + Cr}{Dr^2 + Er + F}$$

Where A, B, C, D, E, F are all nonzero integers, $\gcd(A, B, C, D, E, F) = 1$, and $A > 0$. What is $A + B + C + D + E + F$? You may assume that the only forces present are gravity and the buoyant force from the pool. The airborne ball retains all of its energy as it enters the pool.

Solution 3: The time that it takes the ball released at the surface to reach the bottom of the pool is

$$a = g\left(\frac{\rho_b - \rho_p}{\rho_b}\right) = rg$$

$$t = \sqrt{\frac{2d}{rg}}$$

The time that it takes the ball released above the surface is

$$t_1 = \sqrt{\frac{2h}{g}}$$

$$v_1 = at_1 = \sqrt{2gh}$$

$$v_2 = \sqrt{v_1^2 + 2ad} = \sqrt{2gh + 2rgd}$$

$$t_2 = \frac{v_2 - v_1}{a} = \frac{\sqrt{2gh + 2rgd} - \sqrt{2gh}}{rg}$$

The condition $t_1 + t_2 = t$ must be satisfied

$$\sqrt{\frac{2d}{rg}} = \sqrt{\frac{2h}{g}} + \frac{\sqrt{2gh + 2rgd} - \sqrt{2gh}}{rg} \longrightarrow \frac{\sqrt{2rd}}{r\sqrt{g}} = \frac{r\sqrt{2h} + \sqrt{2h + 2rd} - \sqrt{2h}}{r\sqrt{g}}$$

$$\sqrt{2rd} - (r-1)\sqrt{2h} = \sqrt{2h + 2rd} \longrightarrow 2rd - 4(r-1)\sqrt{rdh} + 2h(r-1)^2 = 2h + 2rd$$

$$h(r-1)^2 - 2(r-1)\sqrt{rdh} = h \longrightarrow 2(r-1)\sqrt{rdh} = h(r-1)^2 - h$$

$$\sqrt{d} = \frac{h(r-1)}{2\sqrt{rh}} - \frac{h}{2\sqrt{rh}(r-1)} \longrightarrow d = \frac{h(r-1)^2}{4r} - \frac{h}{2r} + \frac{h}{4r(r-1)^2}$$

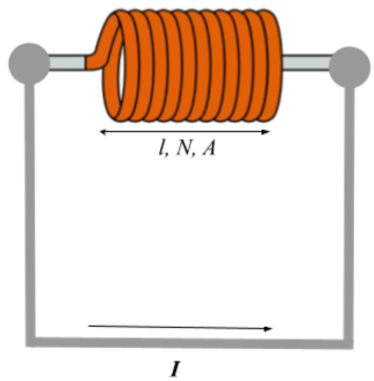
$$d = \frac{(r-1)^4 - 2(r-1)^2 + 1}{4r(r-1)^2} h$$

$$d = \frac{r(r-2)^2}{4(r-1)^2} h = \frac{r^3 - 4r^2 + 4r}{4r^2 - 8r + 4} h$$

The answer is thus $1 - 4 + 4 + 4 - 8 + 4 = \boxed{1}$.

4. Coilfun Consider an air-cored superconducting solenoid of length $l = 1$ m with cross-sectional area $A = 0.1\ \text{m}^2$ and $N = 1000$ turns. We connect the ends of the solenoid to each other with superconducting wires and run a current $I = 1600$ A through the entire setup. Assume that the solenoid behaves ideally.

Cosmonaut Carla has a core with the same dimensions as the solenoid. The core has relative permeability $\frac{\mu_i}{\mu_0} = 10000$ and mass 10 kg. The core is released at rest far from the solenoid and, due to magnetic forces, flies through the solenoid. Cosmonaut Carla can choose to quench the solenoid (instantaneously removing the current) at any time. What is the maximum attainable exit velocity of the core?



Solution 4: The first key idea here is that the magnetic flux through the solenoid $\Phi = LI$ is a conserved quantity, since $\varepsilon = \frac{d\Phi}{dt}$ is necessarily zero to stop current from blowing up. The second key idea is that this is essentially just a conservation-of-energy question, with the decrease in magnetic field energy during the transit of the core through the solenoid being converted to kinetic energy. As a result, the optimal time to shut down the solenoid occurs when magnetic field energy is lowest i.e. when the core is completely inside the solenoid.

Since B is conserved and $\mu \gg \mu_0$, there is essentially no magnetic field energy when the solenoid should be shut down. As a result, we have

$$E_B = \frac{\left(\frac{\mu_0 N I}{L}\right)^2}{2\mu_0} A l = \frac{1}{2} m v^2$$

Plugging in yields $\boxed{v = 179.4 \text{ m/s}}$

5. Into Orbit A cannon is fixed to the top of a platform of height $R = 6.00 \times 10^6$ m, which sits on a planet of mass $M = 6.00 \times 10^{24}$ kg and radius R. Both the cannon and the platform it rests on have negligible mass, and the cannon's chamber is angled horizontally relative to the planet below it. The cannon then fires a cannonball of with mass $m = 45$ kg through a chamber of length $l = 3$ m at a constant acceleration such that the cannonball is able to successfully enter an elliptical orbit around the planet. What is the minimum force that must be applied by the cannon on the cannonball in order to make this possible? You may assume that both the planet and the platform do not move during this process.

Solution 5:
The minimal initial velocity necessary to enter orbit will occur when the semimajor axis is minimized. The semimajor axis is minimized when it is equivalent to $1.5R$.
By the vis-viva equation, the minimum velocity necessary to enter this orbit from the cannon launch is

$$v = \sqrt{GM\left(\frac{2}{2R} - \frac{1}{a}\right)} = \sqrt{GM\left(\frac{1}{R} - \frac{1}{1.5R}\right)} = \sqrt{\frac{GM}{3R}}$$

The cannonball must accelerate to this speed by the time it reaches the end of the barrel. The acceleration can thus be calculated:

$$v_f^2 = v_i^2 + 2a\Delta x$$

$$\left(\sqrt{\frac{GM}{3R}}\right)^2 = 2al \longrightarrow a = \frac{GM}{6Rl}$$

The minimal force is therefore:

$$F = ma = \frac{GMm}{6Rl} \approx \boxed{1.67 \times 10^8 \text{ N}}$$

6. Roundabout 1 A ramp with length d is raised an angle θ ($0° < \theta < 90°$) above the horizontal. A block with mass m is placed at the top of the ramp, with the coefficient of friction between the block and the ramp being μ. Once the block reaches the bottom of the ramp, it retains its velocity as it is smoothly transitioned onto a frictionless circular track with radius d and bank angle θ, rotating on the track without sliding off. A *solution* is a set of values $\{d, \mu, \theta\}$ that result in the situation described above. What is the largest θ (in degrees) for which a solution exists?

Solution 6: The velocity of the block at the bottom of the ramp is

$$f_N = f_g - f_f = mg\sin\theta - \mu mg\cos\theta$$

$$v = \sqrt{\frac{2f_N d}{m}} = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

On the banked ramp we have

$$mg\sin\theta = \frac{mv^2}{d}\cos\theta \longrightarrow gd\sin\theta = v^2\cos\theta \longrightarrow d = \frac{v^2}{g}\cot\theta$$

$$d = \frac{2gd(\sin\theta - \mu\cos\theta)}{g}\cot\theta \longrightarrow \cos\theta(1 - \mu\cot\theta) = \frac{1}{2}$$

By inspection, the largest possible θ is $\boxed{60°}$.

7. Roundabout 2 While the block is going around the circular track, it is given a small push perpendicular to its current velocity and parallel to the surface of the track, causing it to oscillate with period T. What is the smallest possible value of T when $d = 5$ m?

Solution 7: Using central potential, the effective potential of the particle is

$$V(r) = \frac{L^2}{2mr^2}$$

For some small offset δr we can set the following

$$V(d + \delta r) \approx V(d) + \frac{\cos^2\theta}{2}\left(\left.\frac{d^2V}{dr^2}\right|_{r=d}\right)\delta r^2$$

$$\frac{d^2V}{dr^2} = \frac{3L^2}{mr^4}$$

$$\frac{mv^2}{d}\cos\theta = mg\sin\theta \longrightarrow v = \sqrt{gd\tan\theta}$$

$$L = mvd = md\sqrt{gd\tan\theta}$$

$$\frac{d^2V}{dr^2} = \frac{3mg\tan\theta}{d}$$

$$V(d+\delta r) \approx V(d) + \frac{1}{2}\frac{3mg\sin\theta\cos\theta}{d}\delta r^2$$

From this, we retrieve

$$\omega = \sqrt{\frac{3g\sin\theta\cos\theta}{d}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2d}{3g\sin 2\theta}}$$

This is minimized when $\theta = 45°$, so plugging in numbers gives $T_{min} \approx \boxed{3.664 \text{ s}}$

8. Atom Smasher An alpha particle is the nucleus of a ^{4}He atom, and is composed of two protons and two neutrons bound together. A neutron is given speed v and collides with an alpha particle at rest. If all five protons and neutrons become unbound as a result of the collision, what is the minimum possible value of v/c? You may find the following values useful:

$$m_p = 938.27 \text{ MeV}/c^2, \quad m_n = 939.57 \text{ MeV}/c^2, \quad m_\alpha = 3727.4 \text{ MeV}/c^2$$

Solution 8: Let the incoming neutron have Lorentz factor γ. Note that the total energy of the system is $E = (\gamma m_n + m_\alpha)c^2$ and the initial momentum satisfies $p^2 = (\gamma^2 - 1)m_n^2 c^2$. The total energy is minimized if the final protons and neutrons all have the same velocity, in which case the final mass of the system is $M = 2m_p + 3m_n$. Thus, by energy and momentum conservation:

$$E^2 = p^2c^2 + M^2c^4$$

$$(\gamma^2 m_n^2 + 2\gamma m_n m_\alpha + m_\alpha^2)c^4 = (\gamma^2 - 1)m_n^2 c^4 + M^2c^4$$

$$\gamma = \frac{M^2 - m_n^2 - m_\alpha^2}{2m_n m_\alpha} = 1.0378$$

We find $v/c = \sqrt{1 - 1/\gamma^2} = \boxed{0.267}$.

9. Dying Light Follin creates a vat of a peculiar liquid with index of refraction $n = 1 + i(1 \cdot 10^{-6})$. Just a bit complex. While working with the liquid, he accidentally drops a photodetector into it. Follin shines a red laser with wavelength $\lambda = 700$ nm and vacuum intensity $I_0 = 5 \cdot 10^6$ W/m^2 down into the liquid. How far does the photodetector sink before it detects an intensity less than $I_f = 10^{-10}$ W/m^2? Assume that the lab is completely dark and that the laser light is perfectly transmitted into the liquid.

Solution 9: Let the light wave be parameterized as $Ae^{i(kx-\omega t)}$ (i.e the electric field) and let $n = 1 + ia$. Upon entry into the material, observe that $k' = nk$ is the new wave number. Hence the light wave in the fluid is:

$$Ae^{i(nkx-\omega t)} = Ae^{i(kx-\omega t)}e^{-akx}$$

Thus, because intensity is proportional to amplitude squared, we have $I_f = I_0 e^{-2akx}$, giving:

$$x = \frac{1}{2ak}\ln\left(\frac{I_0}{I_f}\right) = \frac{\lambda}{4\pi a}\ln\left(\frac{I_0}{I_f}\right) = \boxed{2.14 \text{ m}}$$

10. Motorized Pendulum 1 A pendulum is made of a massless rod of length $l = 0.5000$ m and a point mass $m = 15.00$ kg hanging at one end. The angle between the rod and the vertical is θ. A motor attached to the pivot supplies a torque. The maximum value of this torque is angle-dependent and is given by $\tau(\theta) = \frac{1+\cos\theta}{2}\tau_0$ for $0 \le \theta \le 90°$.

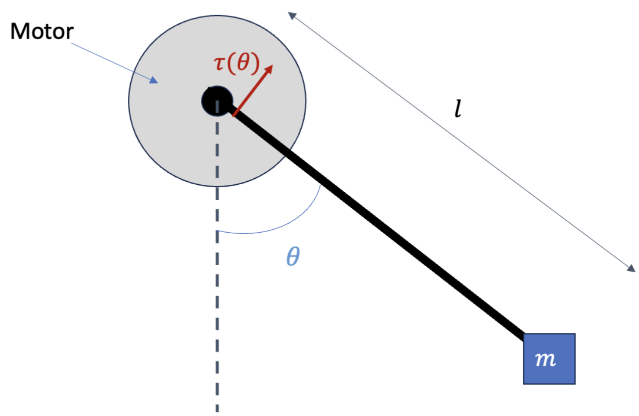


Figure 1: Motorized pendulum

The pendulum initially is given a small angular velocity counterclockwise and is at $\theta = 0$. The mass is extremely sensitive and cannot tolerate high speeds. Therefore, assume the motor always supplies just enough torque for the mass to move at a negligibly small constant speed. What is the minimum value of τ_0 needed so that the pendulum eventually reaches $\theta = 90°$?

Solution 10: The motor has the worst leverage at $\theta = 90°$. Balancing torques,

$$\frac{\tau_0}{2} = mgl$$

The answer is $\boxed{\tau_0 = 2mgl = 147.1 \text{ N} \cdot \text{m}}$.

11. Motorized Pendulum 2 The pendulum initially is at $\theta = 0$. This time, the mass is not so sensitive. The motor may supply its full torque for all θ. What is the minimum value of τ_0 needed so that the pendulum reaches $\theta = 90°$ in a single unidirectional swing?

Solution 11: Our constraint here is that

$$\int_0^t (\tau(\theta) - mgl\sin\theta)\,d\theta \ge 0$$

for all $0 \le t \le 90°$. In other words, the cumulative energy input minus the work done by gravity should never be negative. Carrying out the integral and simplifying a bit, we get the following inequality:

$$\frac{\tau_0}{2mgl} \geq \frac{1-\cos t}{t+\sin t}$$

Between zero and ninety degrees, the function on the left-hand side is monotonically increasing. So, to find its maximum, evaluate it at $t = \pi/2$.

Our answer is $\boxed{\tau_0 = \frac{2}{1+\pi/2} mgl = 57.1 \text{ N} \cdot \text{m}}$.

12. Motorized Pendulum 3 The pendulum initially is at $\theta = 0$. The mass contains extremely sensitive electronics that cannot tolerate speeds above $v_{max} = 0.1000$ m/s. To three significant figures, what is the minimum value of τ_0 needed so that the pendulum reaches $\theta = 90°$ without exceeding this speed threshold?

Solution 12: The optimal route is to accelerate the mass to v_{max}, then as it approaches $\theta = 90°$, let the mass's kinetic energy carry it to the top while the motor fails to provide enough torque to counteract gravity.

We will approximate the answer. Our setup is similar to Motorized Pendulum 1, so the answer to this one should be a small deviation from the answer to that question. Namely, let $\tau_0 = 2mgl - \epsilon$. Near ninety degrees, the torque due to gravity is roughly mgl, and the motor's output torque is roughly $\tau_0 \frac{1+\delta}{2}$, where $\delta = 90° - \theta$ is the distance to the top.

We can find the point at which gravity begins to overpower the motor by setting mgl equal to $\tau_0 \frac{1+\delta}{2}$. This happens at

$$\delta = \epsilon/\tau_0 \approx \epsilon/2mgl$$

Now, from this point onward, the work done by gravity minus the work done by the motor will be positive, causing the mass to slow down. However, this net work should never be larger than the kinetic energy of the mass, $\frac{1}{2}mv^2$. Approximating the torques from gravity and the motor as lines, the net torque is also a line, and we can therefore calculate how much work is done to slow down the mass from the point at which gravity overpowers the motor to ninety degrees. The net torque is

$$mgl - \tau_0 \frac{1+\delta}{2} \approx mgl - 2mgl\frac{1+\delta}{2} = -mgl\delta$$

The area under the graph (i.e. net work) from $\delta = \epsilon/2mgl$ to zero is

$$\frac{1}{2} \cdot \epsilon/2mgl \cdot \epsilon/2 = \frac{\epsilon^2}{8mgl}$$

We can solve for ϵ by setting this area equal to $\frac{1}{2}mv^2$, resulting in $\epsilon \approx 6.6\ Nm$. Therefore, the answer to three significant figures is $\boxed{\tau_0 = 147.1 - 6.6 = 140. \text{ N} \cdot \text{m}}$.

13. Gravitational Oscillations 1 You are given the charge distribution on a conductive ellipsoid described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

If we denote its total charge by q, the surface charge density σ is given by

$$\sigma = \frac{q}{4\pi abc}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}$$

Now suppose we model a planet as a uniform density disk. The issue with this is that everyone at the edges would be pulled towards the center (not "downwards"). In what follows, suppose that the disk has a fixed radius R and a height $h \ll R$. This comes at the cost that different people feel different gravitational "constants" downward. Consider the density distribution of the disk $\rho = \rho(r)$ such that people living on it would only feel a gravitational pull downwards. What is the ratio $\rho(\frac{R}{3})/\rho(\frac{2R}{3})$?

Solution 13:
Here we can use the fact that for a conducting material, the field near the surface is perpendicular to the conductor. First we'll get the charge distribution of a disk from (13), then make an analogy between electrostatics and gravity.
Let $x^2 + y^2 = r^2$, $a = b = R$ and $c \to 0$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$
$$\frac{r^2}{R^2}+\frac{z^2}{c^2} = 1$$
$$\frac{z^2}{c^2} = 1 - \frac{r^2}{R^2}$$

Now σ becomes:

$$\sigma = \frac{q}{4\pi R^2 c}\left(\frac{r^2}{R^4}+\frac{1-\frac{r^2}{R^2}}{c^2}\right)^{-1/2}$$

But $c^{-2} \gg a^{-2}$ so:

$$\sigma \approx \frac{q}{4\pi R^2 c}\left(\frac{1-\frac{r^2}{R^2}}{c^2}\right)^{-1/2}$$

And this gives the charge distribution on a charged disk:

$$\sigma = \frac{q}{4\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{-1/2}$$

Actually, this is the charge distribution on one of the faces of the disk. Because we let $c \to 0$ we have to multiply this by two(you can check that the integral would give half the charge for the formula above).

$$\sigma = \frac{q}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{-1/2}$$

The electric field given by a surface charge σ near the surface is

$$\mathrm{E} = \frac{\sigma}{2\varepsilon_0}$$

And the gravitational field near the surface(also Gauss's law):

$$2\Gamma \mathrm{dS} = 4\pi \mathrm{G}\rho h \mathrm{dS}$$

$$\Gamma = 2\pi \mathrm{G}\rho h$$

So ρh acts similarly to σ. (Γ is the gravitational acceleration)

$$\rho = \rho_0\left(1 - \frac{r^2}{R^2}\right)^{-1/2} \tag{1}$$

Now for the numerical part,

$$\frac{\rho_{\frac{R}{3}}}{\rho_{\frac{2R}{3}}} = \sqrt{\frac{1-\frac{4}{9}}{1-\frac{1}{9}}} = 0.7905694$$

14. Gravitational Oscillations 2 Let the density right at the center of the planet be $\rho_0 = 10000\ \mathrm{kg/m^3}$, and let the height of the disk be $h = 100$ km. What is the mass enclosed in the ring with outer radius $R = 6000$ km and inner radius $R - \epsilon$, where $\epsilon = 1$ km?

Solution 14:
Well, by looking at (2) we can see that ρ goes to infinity as $r \to R$. This doesn't really make sense so let's see if just ignoring a small ring at the outside solves this.

$$\int_{R-\epsilon}^{R} \rho_0 h \frac{2\pi r \mathrm{d}r}{\sqrt{1-\frac{r^2}{R^2}}} = \pi R^2 h \rho_0 \left. \frac{\left(1-\frac{r^2}{R^2}\right)}{\frac{1}{2}} \right|_{R-\epsilon}^{R} =$$
$$= 2\pi R h \rho_0 \sqrt{2R\epsilon}$$
$$= 4.12973 \cdot 10^{21}\ \mathrm{kg}$$

15. Gravitational Oscillations 3 Now imagine that we drill a hole through the disk at a distance $r_0 = \frac{R}{3}$ and drop a ball through. In seconds, what is the period of its oscillation? Neglect air friction.

Solution 15:
Most of the pull came from matter that was not excavated. So we just need the gravitational field at a distance z from the center: We have:

$$\Gamma = 2\pi \mathrm{G}\rho(2 * z)$$

So the motion is described by:

$$\ddot{z} = -4\pi \mathrm{G}\rho z$$

And the angular speed ω of oscillation is $\sqrt{4\pi G\rho_{(r_0)}}$

$$\omega = \sqrt{4\pi G\rho_0}\left(1 - \frac{r_0^2}{R^2}\right)^{-1/4} \tag{2}$$

$$T = \sqrt{\frac{\pi}{G\rho_0}}\left(1 - \frac{r_0^2}{R^2}\right)^{1/4} \tag{3}$$
$$= 2106.6\ \mathrm{s} \tag{4}$$

16. Liquid Lenses A large cylindrical well, with a radius of 1 meter and a depth much greater than the radius ($d \gg r$), is filled with reflective liquid metal. The well and the liquid within it is then set into rotation around its central axis at an angular speed of $\omega = 5$ rad/s, causing the edge of the liquid surface to rise and touch the brim of the well. Positioned directly above the well is a circular lamp with a radius of 1 m, emitting photons vertically downward at a uniform rate and density. If the rate at which photons leave the lamp is r, then the rate at which photons collide with the liquid metal can be expressed as nr, where n is a dimensionless constant. What is the value of n?

Solution 16: Start by looking at a cross section of the liquid through the central axis. Consider a small mass dm of liquid at the surface, a distance x from the axis of rotation. The forces can be balanced as follows:

$$dN \cos\theta = gdm$$

$$dN \sin\theta = x\omega^2 dm$$

$$\tan\theta = \frac{dy}{dx} = \frac{x\omega^2}{g}$$

$$\int dy = \int \frac{x\omega^2}{g} dx \longrightarrow y = \frac{x^2\omega^2}{2g}$$

This equation is parabolic, so the liquid forms a paraboloid lens. The focal point can then be calculated as

$$f = \left(0, \frac{g}{2\omega^2}\right)$$

Because light is incident vertically, photons will either collide with the lens once or twice.
At the brim of the cylindrical well, the height of the liquid relative to its height at the center of the well is

$$\frac{\omega^2}{2g}$$

The points at which incident photons will collide with the lens twice can now be calculated.

$$y = -\left(\frac{\omega^2}{2g} - \frac{g}{2\omega^2}\right)x + \frac{g}{2\omega^2}$$

$$y = \frac{x^2\omega^2}{2g}$$

$$\frac{g}{2\omega^2} - \frac{x^2\omega^2}{2g} = \left(\frac{\omega^2}{2g} - \frac{g}{2\omega^2}\right)x$$

$$\frac{x^2\omega^2}{2g} + \left(\frac{\omega^2}{2g} - \frac{g}{2\omega^2}\right)x - \frac{g}{2\omega^2} = 0$$

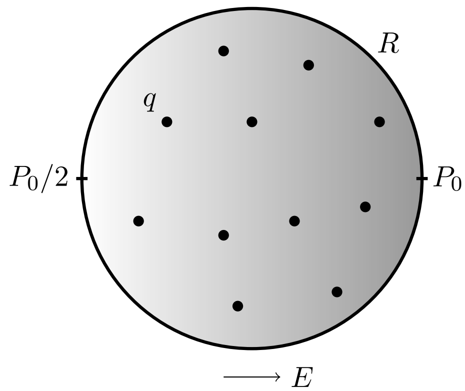
$$x = -1, \frac{g^2}{\omega^4}$$

Any photon initially incident at least $\frac{g^2}{\omega^4}$ away from the central axis will thus hit the mirror twice. The rest of the photons will only hit the mirror once. The value of n can now be calculated.

$$n = 2 - \frac{g^4}{\omega^8} = 2 - \frac{9.8^4}{5^8} \approx \boxed{1.976}$$

17. Electric Slide Consider a gas of small particles, each with charge q, inside an origin-centered spherical chamber of radius R. A uniform electric field $E\hat{\mathbf{x}}$ is applied inside the chamber. The field is adjusted until point $(R, 0, 0)$ has pressure P_0 and point $(-R, 0, 0)$ has pressure $P_0/2$ (at equilibrium).

The electric field is quickly decreased to zero, and the gas comes to equilibrium again. If the final pressure in the chamber is P_1, find P_1/P_0. Neglect interactions between particles and assume that the temperature of the gas remains nearly constant.



Solution 17: When the field is active, the pressure $P(x)$ will be exponential because the gas is in hydrostatic equilibrium. Thus, we have $P(x) = P_0 e^{(x-R)\ln(2)/2}$. Because the temperature remains constant, pressure is proportional to density, so P_1 will be the average of the initial pressure over the volume of the chamber. Letting $R = 1$ and $\frac{\ln(2)}{2} = a$, we have:

$$\begin{aligned} P_1 &= \frac{3}{4\pi R^3} \iiint\limits_V P \, dV \\ &= \frac{3}{4\pi} \int_{-1}^{1} P_0 e^{a(x-1)} \cdot \pi(1 - x^2) \, dx \\ &= \frac{3P_0}{4} \left(e^{a(x-1)} \left(-\frac{x^2}{a} + \frac{2x}{a^2} + \frac{1}{a} - \frac{2}{a^3} \right) \right) \Bigg|_{-1}^{1} \\ &= \frac{3P_0}{4} \left(\left(\frac{2}{a^2} - \frac{2}{a^3} \right) - e^{-2a} \left(-\frac{2}{a^2} - \frac{2}{a^3} \right) \right) \\ &= \boxed{\left(\frac{9}{\ln(2)^2} - \frac{6}{\ln(2)^3} \right) P_0} \approx 0.716 \, P_0 \end{aligned}$$

18. Ping-pong 1 A legal serve in ping-pong requires that the ball bounces on one side of the table and that the ball goes over the net. A certain world-class Olympic ping-pong player does the serve at the level of the table at a distance $d = 1.37$ m from the net of height $h = 15.25$ cm. The Olympic player can give the ball such a spin that the translational speed of the ball is conserved after a bounce but the direction of velocity can be controlled freely. What is the minimal serving speed v_1 (up to two decimal places)?

Solution 18: We first find the optimal speed to reach a point (X, Y) in space from the origin $(0, 0)$. If the launching angle is α and the launch speed v, the kinematic equations for the x- and y-directions are

$$x = vt\cos\alpha \qquad\qquad y = -\frac{1}{2}gt^2 + vt\sin\alpha.$$

Solving for the shape of the trajectory gives

$$y = -\frac{gx^2}{2v^2\cos^2\alpha} + x\tan\alpha.$$

If v is large enough, there exists a respective α for which the point $(x, y) = (X, Y)$ satisfies the equation. Now we note that $1/\cos^2\alpha = 1 + \tan^2\alpha$ which turns the trajectory into a quadratic in $\xi = \tan\alpha$

$$\xi^2 - \frac{2v^2}{gd}\xi + \frac{2v^2 h}{gd^2} + 1 = 0.$$

This has real solutions for ξ if the discriminant is non-negative. If the discriminant is positive, it means there are two angles for the respective speed. This clearly means that the speed is not optimal. Hence we want the discriminant to be zero:

$$\frac{v^4}{g^2d^2} - \frac{2v^2h}{gd^2} - 1 = 0.$$

This is a biquadratic in v and solving:

$$v^2 = gY + g\sqrt{X^2 + Y^2} \implies v_0 = \sqrt{gY + g\sqrt{X^2 + Y^2}}.$$

Now, let's get back to the problem at hand. Let x be the distance from the edge of the table at the bouncing point. As the speed is conserved during the bounce, the minimal speed must be such that at the point x the ball can reach both the edge of the table (kinematic trajectories are reversible) and the top of the net (the ball has to go over the net). Using the derived result

$$v(x) = \max\left\{\sqrt{gh + g\sqrt{h^2 + (d-x)^2}}, \sqrt{gx}\right\}.$$

Clearly the first term increases with x and the other term decreases with x. Thus the global minimum of $v(x)$ is found at the point where the two terms are equal. This corresponds to

$$h + \sqrt{h^2 + (d-x)^2} = x \implies x = \frac{d^2}{2(d-h)}$$

and thus

$$v_1 = d\sqrt{\frac{g}{2(d-h)}} \approx 2.75 \text{ m/s}.$$

19. Ping-pong 2 We consider a serve with n bounces before going over the net. The Olympic player is so incredibly good that he can control the direction of the velocity after each bounce as he pleases. Naturally more bounces decreases the minimal serving speed v_n. However, for some N, when $n \geq N$ the minimal serving speed no longer decreases if bounces are added, i.e. N is the smallest natural number such that $v_m = v_N$ for all $m \geq N$. Find v_{N-1}^N.

Solution 19: We can generalise the argument of the last problem by noting that

$$v_n \geq \sqrt{gh + g\sqrt{h^2 + x^2}},$$

where x is the distance of the last bounce from the net. Thus we wish to get the ball as close to the net as possible in n bounces. This happens when all the previous bouncers happen at an angle 45°. The range of one of these bounces is $\ell = \frac{v^2}{g}$ as one can find from the trajectory equation. Thus we have that the distance of the final bounce from the net is:

$$x = d - n\ell.$$

We also note that $v_n \geq \sqrt{2gh}$ based on the lower limit of the previous equation ($x = 0$). So if we reach the net in less than n bounces and thus get to $x = 0$ for the last bounce, the minimal speed will still be v_n. I.e. when $d - n\ell < 0 \implies n > \frac{d}{2h}$ the minimal speed will be

$$v_n = \sqrt{2gh}.$$

As $d/2h \approx 4.5$, $N = 5$.

Now if $n < N$, we won't reach the net in n bounces, and the distance from the net for the last bounce is $x = d - n\ell$. I.e. the speed to get the ball's last bounce to be a distance x away will be $v = \sqrt{g(d-x)/n}$. Similarly to the previous problems, this means that the optimal speed is achieved at the x for which the optimal speed to get there is the same as to go over the net from there. I.e. we get that

$$h + \sqrt{h^2 + x^2} = \frac{d - x}{n},$$

which yields the quadratic equation

$$(n^2 - 1)x^2 + 2(d - nh)x + 2ndh - d^2 = 0,$$

from which we get

$$x = \frac{nh - d \pm n\sqrt{h^2 + d^2 - 2ndh}}{n^2 - 1}.$$

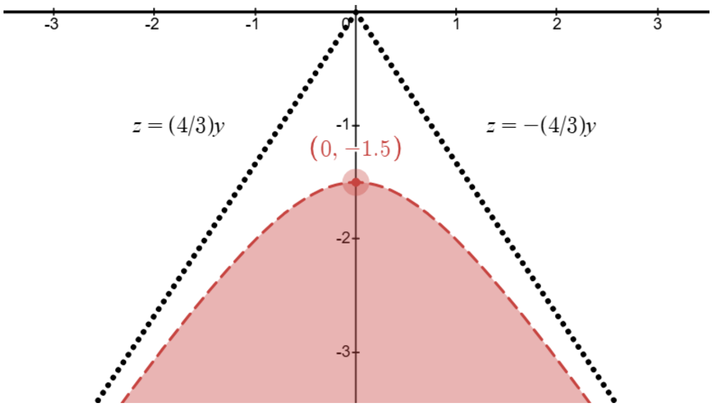
As $d > 2nh$ the smaller root is negative and thus non-physical. Hence, we take the positive root. Substituting this in one of the expressions for v we get

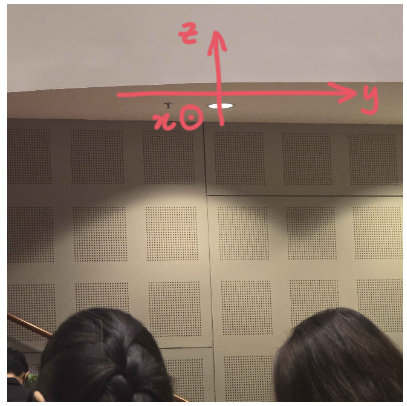
$$v_n = \sqrt{\frac{g}{n^2 - 1}\left(nd - h - \sqrt{h^2 + d^2 - 2ndh}\right)}.$$

Thus

$$v_{N-1}^N = v_4^5 \approx \boxed{17.9\ (\text{m/s})^5.}$$

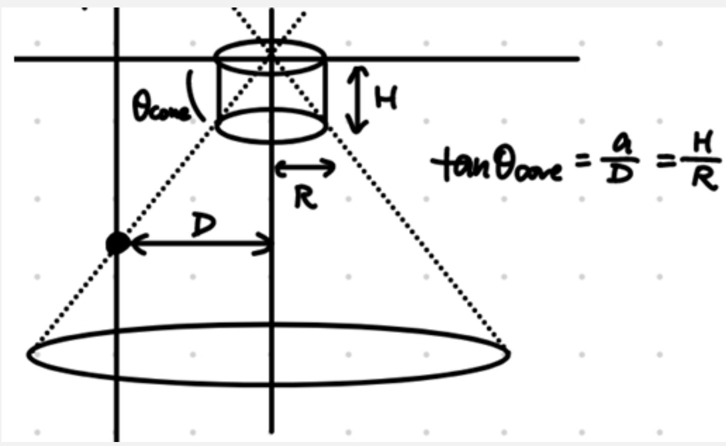
20. An Envelope of Light A point light source on the ceiling is located at the center of a cylindrical housing (with an open base) of radius R and height H. A wall is a horizontal distance D from the center of the cylinder. Now consider a coordinate system with the light source at the origin. The wall, at $x = -D$, has the following shape in the $y - z$ plane:

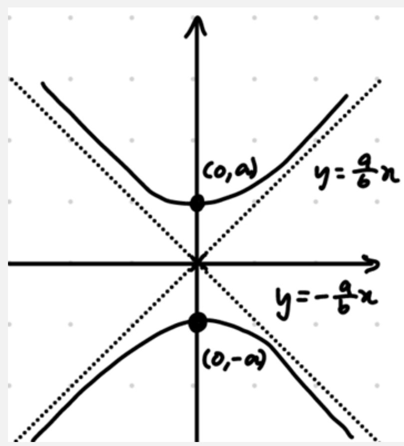




The vertical coordinate of the highest point of the curve observed is -1.5 m, while the gradients of the lines asymptotically tangent to the curve are $\pm 4/3$. On the right is shown an example setup of this phenomenon. Find the horizontal distance D of the wall to the light source.

Solution 20: The point light source emits light with spherical symmetry, but because of the cylindrical housing what is interpreted is a light "cone" instead. This cone intersecting the wall at $x = -D$ is what gives rise to the hyperbola shown.





By virtue of conic sections having constant eccentricity, it can be expressed as: $e = \sin\theta_{\text{plane}}/\sin\theta_{\text{cone}} = 1/\sin\theta_{\text{cone}}$ since the wall has $\theta = 90°$. Alternatively, one can also express eccentricity by the formula $e = c/a = \sqrt{a^2+b^2}/a$ where $c = \sqrt{a^2+b^2}$ is the distance from the origin to the focus. Rearranging this gives us:

$$e = \frac{1}{\sin\theta_{\text{cone}}} = \sqrt{1+\left(\frac{b}{a}\right)^2} \implies \tan\theta_{\text{cone}} = \frac{a}{b} = 1.33$$

This can then be related to the horizontal distance D by $\tan\theta_{\text{cone}} = a/D$ to give $D = b = 1.5\text{m}/(4/3) = \boxed{1.125 \text{ m}}$.

21. Under The Lamplight A large vat of the magical liquid Ophonium lies before you, with a depth of 5 meters. This liquid has a special property - its index of refraction changes variable to its depth! Its index of refraction can be expressed by the equation

$$n = 1 + 2y$$

where y is the depth of the liquid, in meters. A lamp, acting as a point source of light, is hung 3 meters above the vat. Light emanates from the lamp, casting its glow onto a circular section of the Ophonium's surface directly beneath it, covering an area of 3π square meters. As this light continues to travel downward, it enters the Ophonium, gradually penetrating its depths until it reaches the bottom of the vat. What is the area, in square meters, of the circle illuminated at the bottom of the vat? You may find the following integral useful:

$$\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1}(x) + C$$

Solution 21: The circle below the lamp has a radius of

$$r = \sqrt{\frac{3\pi}{\pi}} = \sqrt{3}$$

The greatest angle that the light travels from the vertical is therefore

$$\theta = \arctan \frac{\sqrt{3}}{3} = 30^\circ$$

We can examine a cross section of the light's path. Consider Snell's Law, $n_1 \sin\theta = n_2 \sin\theta$. It can be seen that at any point along the light's path through the liquid, $n \sin\theta$ will always equal some constant value. In this case, because the incident angle is 30 degrees and the index of refraction of air is 1, this constant value will be $\sin 30^\circ = 0.5$. We can assign the light ray the initial conditions $(x, y) = (\sqrt{3}, 0)$, as the light enters the Ophonium a horizontal distance of $\sqrt{3}$ from the lamp.
We have:

$$n \sin\theta = \frac{1}{2} \longrightarrow \sin\theta = \frac{1}{2n} = \frac{1}{2+4y}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2+4y} \longrightarrow \frac{1}{\sqrt{1+\tan^2\left(\frac{\pi}{2}-\theta\right)}} = \frac{1}{2+4y}$$

$$\frac{1}{\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}} = \frac{1}{2+4y} \longrightarrow \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} = 2+4y$$

$$\frac{\delta y}{\delta x} = \sqrt{(2+4y)^2-1} \longrightarrow \frac{\delta x}{\delta y} = \frac{1}{\sqrt{(2+4y)^2-1}}$$

$$\int \delta x = \int \frac{1}{\sqrt{(2+4y)^2-1}} \delta y$$

$$x = \frac{1}{4}\cosh^{-1}(2+4y) + C \longrightarrow x = \frac{1}{4}\left(\cosh^{-1}(2+4y) + 4\sqrt{3} - \ln\left(2+\sqrt{3}\right)\right)$$

Plugging in $y = 5$ gives $x \approx 2.3487$. This is the radius of the circle formed at the bottom of the vat. Thus, the total area is

$$x^2\pi \approx \boxed{17.33 \text{ m}^2}$$

22. Cannonball On flat ground, a cannonball is shot with a speed of $v = 13.3$ m/s over a wall of height $h = 5$ m located a distance $d = 10$ m away from the cannon. If $x_{\max}$ is the farthest away the ball can be shot over the wall and $x_{\min}$ is the shortest distance the ball can reach while going over the wall, find $x_{\max} - x_{\min}$.

Solution 22: Clearly (as $d > h$) if v is high enough, the optimal trajectory to reach $x_{\max}$ is achieved when the launch angle $\alpha = 45°$. Thus we have to first determine whether or not v is high enough.

If $\alpha = 45°$ corresponds to the optimal trajectory over the wall, the highest point of the wall (d, h) is at or under the trajectory. The equation of the trajectory is well-known and easy to derive:

$$y = -\frac{gx^2}{2v^2\cos^2\alpha} + x\tan\alpha$$

so $\alpha = 45°$ is optimal if

$$h \le -\frac{gd^2}{v^2} + d \implies v \ge d\sqrt{\frac{g}{d-h}} = 14 \text{ m/s},$$

which is not true. Thus $\alpha = 45°$ is not optimal.

Now clearly the ball cannot go over the wall if $\alpha \le 45°$. Thus we have to increase the angle which decreases the range as the range is a decreasing function of α when $\alpha > 45°$ which can be seen from the range equation (comes directly from the trajectory equation above):

$$R = \frac{2v^2\sin\alpha\cos\alpha}{g} = \frac{v^2\sin 2\alpha}{g}.$$

Thus the optimal trajectory now is the one that just barely touches the top of the wall for the minimal α. I.e. we wish (d, h) to be on the trajectory. If we plug in $(x, y) = (d, h)$ to the trajectory equation and use $1/\cos^2\alpha = 1 + \tan^2\alpha$ and solve for $\xi \equiv \tan\alpha$ from the quadratic equation that ensues we get two possible roots

$$\xi_\pm = \frac{v^2}{gd} \pm \sqrt{\frac{v^4}{g^2d^2} - \frac{2v^2h}{gd^2} - 1}.$$

As $\xi = \tan\alpha$ and tan is an increasing function (in our range), we are interested in the smaller root. Thus

$$\alpha = \arctan\xi_-.$$

Thus the range equation gives us

$$x_{\max} = \frac{2\xi_-}{1+\xi_-^2}\frac{v^2}{g}.$$

The argument for the minimal distance is essentially the same. The top of the wall still has to be under the trajectory and $\alpha > 45°$ so now we want the maximal α which corresponds to ξ_+. Thus

$$x_{\min} = \frac{2\xi_+}{1+\xi_+^2}\frac{v^2}{g}.$$

And hence

$$x_{\max} - x_{\min} = \frac{2v^2}{g}\left(\frac{\xi_-}{1+\xi_-^2} - \frac{\xi_+}{1+\xi_+^2}\right) \approx \boxed{5.38 \text{ m}}.$$

Note for a full solution (not necessary to get the numerical answer) one should also check if the launch speed given is even high enough to go over the wall at all. This minimal speed is relatively easy to derive (especially using the envelope curve):

$$v_{\text{min}} = \sqrt{gh + g\sqrt{h^2 + d^2}} \approx 12.6 \text{ m/s} < v,$$

so the question is well-posed.

23. Filament 1 Follin has developed a revolutionary new lightbulb filament. The filament is in the shape of a spherical shell of radius 3 cm and thickness 0.5 mm, and the ends of the filament are diametrically opposite. To attach the wires, the filament is slightly flattened on the ends, forming two circles of radius 0.01 mm, each of which is completely covered by the corresponding contact wire. If the resistivity of the filament material is 0.050 $\Omega \cdot$ m, what is the resistance of Follin's lightbulb?.

Solution 23:
Orient the sphere so its ends are vertical and align on the z axis. The equipotential surfaces are rings with planes parallel to the xy plane. The ring element at polar angle ϕ has length (along direction of current) $rd\phi$ and cross section area $2\pi r \sin\phi \cdot t$ (note that $r \gg t$). Integrating from angle $\phi_i = \frac{0.01}{30}$ to $\phi_f = \pi - \frac{0.01}{30}$, the total resistance of the filament is

$$R = \int_{\phi_i}^{\phi_f} \frac{\rho \cdot r d\phi}{2\pi r \sin\phi \cdot t} = \boxed{277\ \Omega}.$$

24. Filament 2 Follin now modifies his filament to have six instead of two flattened circles arranged symmetrically around the sphere. He places contact wires at two flattened circles that are spaced 90° apart on their shared great circle. Find the new resistance of the lightbulb.

Solution 24:
Let the two contact points be A and B. We use a superposition argument. Consider the following two scenarios:

- Inject current I into A and draw out current I uniformly from the surface of the sphere. The current passing through the shell at polar angle ϕ is equal to the current leaving the sphere for all polar angles greater than ϕ, which is $\frac{1+\cos\phi}{2} I$ by consideration of surface area. Then the current density at ϕ is $J = \frac{\frac{1+\cos\phi}{2} I}{2\pi r \sin\phi \cdot t}$. Letting $\phi_i = \frac{0.01}{30}$ and $\phi_f = \frac{\pi}{2} - \frac{0.01}{30}$, the voltage between A and B is

$$V = \int \mathbf{E} \cdot d\mathbf{l} = \int_{\phi_i}^{\phi_f} \rho J \cdot r d\phi = \int_{\phi_i}^{\phi_f} \frac{\rho I(1+\cos\phi)}{4\pi t \sin\phi} d\phi.$$

- Draw out current I from B and inject current I uniformly into the surface of the sphere. We obtain the same voltage between A and B as in the other scenario.

Superposing the two scenarios, we have a current distribution that injects current I into A and draws out current I from B, with voltage difference $2V$ between them. Thus, the effective resistance between A and B is $\frac{2V}{I} = 2\int_{\phi_i}^{\phi_f} \frac{\rho(1+\cos\phi)}{4\pi t \sin\phi} d\phi = \boxed{266\ \Omega}$.

25. A Question of Counting A uniform beam of cold neutrons (neutrons have mass $m = 1.67 \times 10^{-27}$ kg and speed $v = 0.66$ m/s) passes through a thin slit of width $d = 0.5$ mm. A detector, of width $w = 3$ cm, is placed a distance $r = 15$ m from the slit. Find the percentage of neutrons passing through the slit that are actually recorded by the detector.

Solution 25: The key idea is to realize that the 1D probability density function is analogous to the intensity from single slit diffraction. More precisely, let us consider the 1D probability density at a given position x in the plane of the detector. $P = \|\Psi_x\|^2 = \Psi_x^* \Psi_x$ corresponds to taking the square of the norm of summed phasors, which is analogous to optical intensity being the square of the norm of the electric field phasor.

This makes our life easier, because we can simply reuse the derivations from classical wave optics! The DeBroglie wavelength of the neutrons is

$$\lambda = \frac{h}{p} = \frac{h}{mv} = 601.1 \text{ nm}$$

As can be derived using phasors (or simply by reusing the analogous formula from wave optics), we have

$$\|\Psi_x\|^2 \propto \left[\frac{\sin\left(\frac{\pi d \frac{x}{\sqrt{r^2+x^2}}}{\lambda} \right)}{\left(\frac{\pi d \frac{x}{\sqrt{r^2+x^2}}}{\lambda} \right)} \right]^2$$

We have to be careful to normalize the probability density function when calculating the final result:

$$P = \frac{\int_{-w/2}^{w/2} \|\Psi_x\|^2 \, dx}{\int_{-\infty}^{\infty} \|\Psi_x\|^2 \, dx}$$

Plugging in the given parameters, this gives us a final percentage of

$$P \times 100\% = \boxed{90.2\%}$$

26. Peculiar Pump A box with volume $V = 1$ m^3 and initial temperature $T_0 = 100$ K contains monatomic ideal gas initially kept at a constant low pressure $p_1 = 100$ Pa. The mass of the gas molecules is $m = 7 \times 10^{-27}$ kg. The box is connected to a very large reservoir which contains monatomic ideal gas at temperature $T_2 = 400$ K and pressure $p_2 > p_1$. A small hole with area $A = 10^{-8}$ m^2 is poked in the box at time $t = 0$ days, allowing gas to escape. (The box is no longer at constant pressure after this point.) For every molecule of gas that escapes through this hole, 3 molecules of gas are let into the box from the reservoir. (The particles let in from the reservoir are selected with probability proportional to their velocity component perpendicular to the hole. In other words, the particles effuse through the hole but with the door to the hole restricting their number to 3 times the number of outgoing particles.) How many days does it take for the temperature in the box to double? Assume changes in the reservoir's pressure and temperature are negligible. For reference, the rate of particles escaping out of a hole with area A is given by:

$$\Phi = \frac{pA}{\sqrt{2\pi m k_B T}}$$

where p is the pressure of the gas, m is the mass of the gas molecules, k_B is the Boltzmann constant and T is the temperature of the gas. Please note that the average energy per particle of particles leaving via effusion is $2kT$.

Solution 26: Let N be the number of particles in the box as a function of time, and let T_1 be the temperature of the box as a function of time. Then effusion gives

$$\frac{dN}{dt} = \frac{2AN}{V}\sqrt{\frac{k_BT_1}{2\pi m}} \tag{5}$$

Typically there would be no factor of 2 and $\frac{dN}{dt}$ would be negative, but 3 new particles enter the box from the reservoir every time one leaves from the hole.

We also know that energy leaves the box through the hole through the thermal energy of the leaving gas. A similar process happens to the gas entering from the reservoir. It can be shown that the average energy of a particle effusing out of a small hole is $2kT$, and since the particles coming in from the reservoir are also effusing in, this is the average energy of the incoming and outgoing particles. The rate of gas leaving is $\frac{1}{2}\frac{dN}{dt}$ and the rate of gas entering is $\frac{3}{2}\frac{dN}{dt}$, so

$$\frac{d}{dt}\left(\frac{3}{2}Nk_BT_1\right) = \left(\frac{3}{2}\frac{dN}{dt}\right)(2k_BT_2) - \left(\frac{1}{2}\frac{dN}{dt}\right)(2k_BT_1) \tag{6}$$

Simplifying yields

$$\frac{dT_1}{dt} = \frac{1}{N}\frac{dN}{dt}\left(2T_2 - \frac{5}{3}T_1\right) \tag{7}$$

Plugging in for dN/dt yields

$$\frac{dT_1}{dt} = \frac{A}{V}\sqrt{\frac{k_BT_1}{2\pi m}}\left(4T_2 - \frac{10}{3}T_1\right) \tag{8}$$

Integrating this yields

$$t = (0.007524)\frac{V}{A}\sqrt{\frac{2\pi m}{k_B}} \tag{9}$$

Which, plugging in the numbers, is 0.492 days.

27. Air Cushion An air cushion is in the shape of a cylinder with length $\ell = 10.0$ m and circular cross-sectional radius $R = 28$ cm. The ends of the cylinder lie in vertical planes, and the length lies parallel to the horizontal ground. It is filled with an incompressible gas. Both the surface of the air cushion and gas inside have negligible weight compared to other forces in the scenario. The surface maintains a a constant surface tension $\gamma = 5.0$ N/m whenever deformed. A flat slab of mass $m = 12.0$ kg which is wider than the cushion is balanced on top, squishing the cushion. Find the new horizontal width of the cushion. Assume that its cross-section remains symmetric about a vertical axis.

Solution 27:
When the cushion is squished, since the surface tension of the surface stays the same and the outline of the cushion must be smooth, the sides of the cushion not in contact with the ground or slab form semicircles. Let the radius of the semicircles be r. Since the surface maintains a constant surface tension, the gas inside the cushion has an excess pressure of $\Delta P = \frac{\gamma}{r}$ over the outside by Young-Laplace. Let the width of the flat portion of the surface in contact with the ground be b and the flat portion of the surface in contact with the slab be c.
Let N be the normal force on the cushion from the ground. Force balance gives the following equations:

- System of the slab and cushion: $N = mg$.
- Top flat portion of the cushion: $N = c\ell\Delta P$.
- Bottom flat portion of the cushion: $N = b\ell\Delta P$.

Thus we have $b = c = \frac{mg}{\ell\Delta P}$. Volume conservation preserves the area of the cross-section, so we have

$$\begin{aligned}\pi R^2 &= c \cdot 2r + \pi r^2 \\ &= (\pi + \frac{mg}{\gamma\ell})r^2 \\ \implies r &= R\sqrt{\frac{\pi}{\pi + \frac{2mg}{\gamma\ell}}}.\end{aligned}$$

The final width is

$$\begin{aligned}w &= c + 2r \\ &= (\frac{mg}{\gamma\ell} + 2)r \\ &= \boxed{0.771 \text{ m}}.\end{aligned}$$

28. Soapy oscillator On a smooth table lies a square frame made of four homogeneous rods of length $\ell = 50$ cm and mass $m = 150$ g which are hinged together at the corners. Each (massless) hinge carries a charge $q = 1.5 \cdot 10^{-6}$ C. Before the frame was put on the table, it was put in a liquid soap mixture which left a soap film within the frame defined by the rods. The surface tension of said soapy water is $\sigma = 0.035$ N/m. It turns out that it is possible to have small amplitude oscillations where opposing corners of the frame have opposite velocities (away/towards from the centre). What is the respective angular frequency of the oscillations?

Solution 28: Let the angle of the sides change by θ. The kinetic energy can be written as

$$K = 4 \cdot \frac{1}{2}\left(\frac{1}{12}m\ell^2 + m\frac{\ell^2}{4}\right)\dot{\theta}^2 = \frac{2}{3}m\ell^2\dot{\theta}^2.$$

The potential energy associated with the soap film is simply

$$U_\sigma = 2\sigma\ell^2\cos(2\theta) \simeq 2\sigma\ell^2\left(1 - 2\theta^2\right).$$

The potential energy associated with the positions of the charges is simply

$$U_\varepsilon = \frac{kq^2}{\ell}\left(4 + \frac{1}{2\sin(\pi/4 - \theta)} + \frac{1}{2\sin(\pi/4 + \theta)}\right) \simeq \frac{kq^2}{\ell}\left(4 + \sqrt{2} + \frac{3\sqrt{2}}{2}\theta^2\right).$$

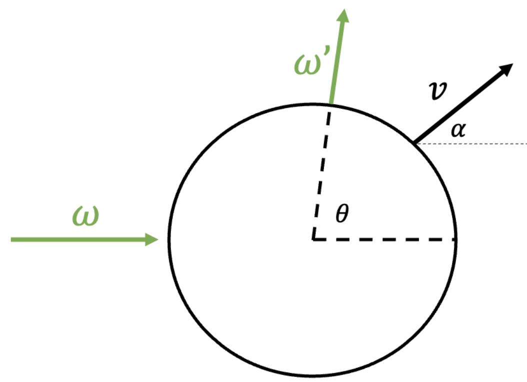
Total energy is conserved and thus $\dot{E} = 0$. Differentiating the total energy and regrouping the terms thus gives

$$\ddot{\theta} + \frac{9\sqrt{2}\frac{kq^2}{\ell} - 24\sigma\ell^2}{4m\ell^2}\theta = 0.$$

Thus

$$\omega = \sqrt{\frac{9\sqrt{2}\frac{kq^2}{\ell} - 24\sigma\ell^2}{4m\ell^2}} \approx \boxed{1.43 \text{ rad/s.}}$$

29. Relativistic Scattering A small spherical particle traveling at a speed $v = 0.5c$ at an angle $\alpha = 45°$ from the horizontal is struck by an electromagnetic plane wave of angular frequency $\omega = 7.08 \times 10^{15}$ Hz propagating directly to the right. In its own reference frame, the particle scatters light in all directions with the same frequency as the frequency of incident light it perceives. Due to the relativistic Doppler effect, however, the frequency of the scattered light measured in the lab frame is generally not the same as the incident light frequency. What is the angular frequency ω' of light scattered into a scattering angle of $\theta = 89°$? Assume the radius of the particle R is small enough that $R\omega \ll c$.



Solution 29:
We first transform into the reference frame of the particle. In this frame, it becomes a stationary particle receiving and emitting light isotropically. What frequency of light, we ask? We will perform a Lorentz transformation to find out. Let the x-axis be along the direction of the particle velocity. Setting $c = 1$, the wavevector 4-vector in the lab frame is

$$k^\mu = (\omega, \omega \cos\alpha, -\omega \sin\alpha)$$

Transforming to the particle frame, the new 4-vector is

$$k_1^\mu = (\omega\gamma(1 - v\cos\alpha), ..., ...)$$

The x- and y- components do not matter as we only care about the t-component, which tells us the frequency of the light in the particle frame. So light of frequency $\omega_1 = \omega\gamma(1 - v\cos\alpha)$ gets scattered into all directions. Consider light that gets scattered into an arbitrary angle ϕ from the x-axis. Its 4-vector would be

$$k_2^\mu = (\omega_1, \omega_1 \cos\phi, \omega_1 \sin\phi)$$

Transforming this back into the lab frame, the final 4-vector is

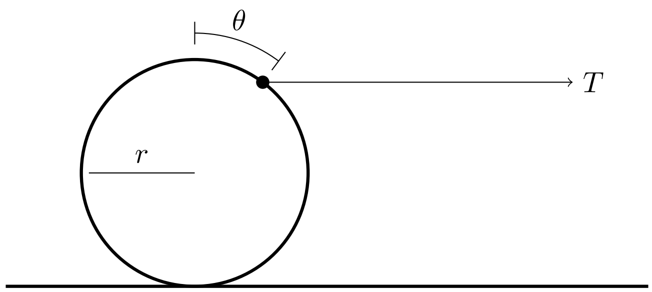
$$k_3^\mu = (\omega_1\gamma(1 + v\cos\phi), \omega_1\gamma(\cos\phi + v), \omega_1 \sin\phi)$$

In the lab frame, the angle δ between the x-axis and the direction of propagation of k_3 is given by

$$\tan\delta = \frac{\sin\phi}{\gamma(\cos\phi + v)}$$

We can start plugging in numbers. We find that $\gamma = 1.155$, $\omega_1 = 5.29 \cdot 10^{15}$ Hz, and $\tan\delta = 0.966$. We can solve for ϕ, getting back $\phi = 70.0°$.
Our answer is therefore $\boxed{\omega' = k_3^0 = 7.15 \cdot 10^{15} \text{ Hz}}$.

30. Reluctant Roller A hoop of mass m and radius r rests on a surface with coefficient of friction μ. At time $t = 0$, a string is attached to the hoop's highest point and a constant horizontal tension T is applied. By time t, the hoop has rotated by angle $\theta(t)$. What is the minimum value of $\frac{T}{\mu mg}$ such that $\theta(t)$ has a local maximum (i.e., is not strictly increasing)? You may need to graph an implicit function.



Solution 30: The motion can be divided into two regimes: rolling-without slipping, where the frictional force is large enough to counteract the tension, and rolling-with-slipping, which slows the rotation until θ reaches a maximum. First, let's consider the period where the hoop rolls without slipping. If F_f is the magnitude of the frictional force, we have:

$$\ddot{x} = r\ddot{\theta} \Longrightarrow \frac{T - F_f}{m} = \frac{Tr\cos\theta + F_f r}{mr^2} \Longrightarrow F_f = \frac{T(1-\cos\theta)}{2}$$

Because $F_f \leq \mu mg$, rolling without slipping ends at the critical angle $\theta_c = \cos^{-1}\left(1 - \frac{2\mu mg}{T}\right)$. To fully define the initial conditions, we must also find $\dot{\theta}$ at the critical point. Consider the equation of motion for θ during rolling-without-slipping:

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{Tr\cos\theta + F_f r}{mr^2} = \frac{T(1+\cos\theta)}{2mr}$$

$$\Rightarrow \int_0^{\theta_c} \dot{\theta}\ d\dot{\theta} = \int_0^{\theta_c} \frac{T(1+\cos(\theta))}{2mr}\ d\theta \Rightarrow \dot{\theta}_c^2 = \frac{T(\theta_c + \sin(\theta_c))}{mr}$$

Now, consider the rolling-with-slipping period. Letting θ_m be the maximum value of θ reached, we will use the same trick to compute $\dot{\theta}_m$:

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{Tr\cos\theta + F_f r}{mr^2} = \frac{T\cos(\theta)}{mr} + \frac{\mu g}{r}$$

$$\Rightarrow \int_{\theta_c}^{\theta_m} \dot{\theta}\ d\dot{\theta} = \int_{\theta_c}^{\theta_m} \frac{T\cos(\theta)}{mr} + \frac{\mu g}{r}\ d\theta \Rightarrow \frac{\dot{\theta}_m^2}{2} - \frac{\dot{\theta}_c^2}{2} = \frac{\mu g(\theta_m - \theta_c)}{r} + \frac{T(\sin(\theta_m) - \sin(\theta_c))}{mr}$$

$$\Rightarrow \dot{\theta}_m^2 = \frac{2\mu g(\theta_m - \theta_c)}{r} + \frac{T(2\sin(\theta_m) + \theta_c - \sin(\theta_c))}{mr}$$

For the hoop to stop at θ_m, we must have $\dot{\theta}_m = 0$. Thus:

$$\frac{mr}{T}\dot{\theta}_m^2 = \frac{2\mu mg(\theta_m - \theta_c)}{T} + 2\sin(\theta_m) + \theta_c - \sin(\theta_c) = 0$$

Letting $a = \frac{T}{\mu mg}$:

$$\Rightarrow \frac{2(\theta_m - \cos^{-1}(1-2/a))}{a} + 2\sin(\theta_m) + \cos^{-1}(1-2/a) - \sin\big(\cos^{-1}(1-2/a)\big) = 0$$

$$\Rightarrow \theta_m + a\sin(\theta_m) + (a/2-1)\cos^{-1}(1-2/a) - \sqrt{a-1} = 0$$

Next, we graph this implicit function of θ_m and a (taking care to limit ourselves to the physically relevant region where $\theta_m > \theta_c$). We find that the minimum value of a over all solutions is $\boxed{3.888}$. The equivalent condition $\frac{da}{d\theta_m} = 0$ can be used to find the solution without graphing.

31. Solenoid Spring A solenoid also functions as a spring of spring constant $k = 50$ N/m. It is connected in series with a resistor of very small resistance. One end of the solenoid is firmly fixed to its wire while the other end has a conducting ring of mass $m = 0.25$ kg that is free to frictionlessly slide on its wire but always remains in contact. The inductance is $L_0 = 3.0$ mH at its relaxed length $\ell_0 = 20$ cm. If the free end of the solenoid is slightly displaced, find the angular frequency of the resulting oscillations. The initial current through the circuit is $I = 8$ A. Assume that the thermal power lost by the resistor is negligible.

Solution 31:
Since the resistance is small, the flux $\Phi = LI$ remains constant. The given inductance can be written as $L_0 = \frac{\mu_0 N^2 \pi r^2}{\ell_0}$, where N is the number of turns and r is the radius. Displace the ring by a small distance x. The inductance of the solenoid at this moment is

$$L = \frac{\mu_0 N^2 \pi r^2}{\ell_0 + x} = \frac{L_0}{1 + x/\ell_0},$$

so

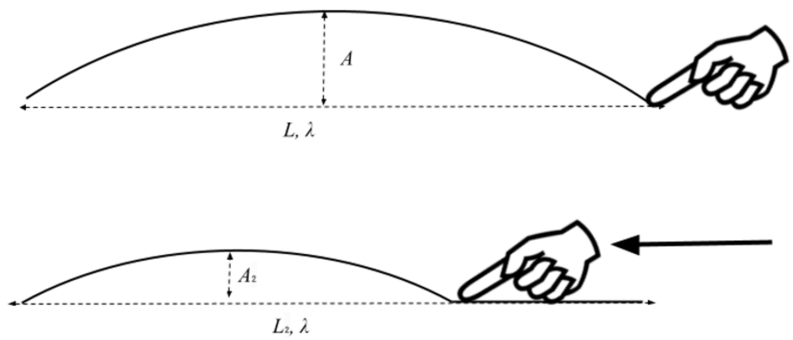
$$I = I_0(1 + \frac{x}{\ell_0}).$$

The energy of the system is $E = \frac{1}{2}LI^2 + \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$. Since the power loss through the resistor is small, we have

$$0 = \frac{dE}{dt} = L_0 I_0 \dot{I} + kx\dot{x} + m\dot{x}\ddot{x}$$
$$0 = L_0 I_0^2/\ell_0 + kx + m\ddot{x}.$$

From here we read off angular frequency $\omega = \sqrt{\frac{k}{m}} = \boxed{14.1 \text{ rad/s}}$.

32. Devil's Trill Consider a string of length $L = 1$ m and linear mass density $\lambda = 1$ g/m, fixed on both ends and vibrating in the first normal mode with amplitude $A = 0.125$ cm. A frictionless, negligibly-small finger, initially at the right endpoint, slides slowly toward the left, flattening the oscillation as it goes. When the vibrating part of the string has length $L_2 = 12.5$ cm, find the new amplitude of vibrations A_2, in meters. You may assume $A_2 \ll L_2$. Diagrams are not necessarily drawn to scale.



Solution 32: Let the string have tension T, and let the displacement be $A\sin(\pi x/L)\sin(\omega t)$. We compute the total energy by finding the length at the peak of an oscillation:

$$\begin{aligned}E &= T(L_{\text{peak}} - L)\\ &= T\left(\int_0^L \sqrt{1+\frac{\pi^2A^2}{L^2}\cos^2\frac{\pi x}{L}}\,dx - L\right)\\ &\approx T\left(\int_0^L 1+\frac{\pi^2A^2}{2L^2}\cos^2\frac{\pi x}{L}\,dx - L\right)\\ &= \frac{\pi^2A^2T}{4L}\end{aligned}$$

At time t, the net horizontal force on the finger is:

$$F_x = T\left(1-\cos\left(\arctan\left(\frac{A\pi}{L}\sin(\omega t)\right)\right)\right) \approx \frac{\pi^2A^2T}{2L^2}\sin^2(\omega t)$$

Thus, the average horizontal force on the finger is $\langle F_x\rangle = -\frac{dE}{dL} = \frac{\pi^2A^2T}{4L^2} = \frac{E}{L}$. This gives $E \propto \frac{1}{L}$, so A is constant, and $A_2 = A = \boxed{1.25\times10^{-3}\text{ m}}$.

Note that the question could also have been solved through the adiabatic invariant, or alternately by realizing that the restoring force is linear in displacement at all points on the string. As a result, a valid solution, by superposition, is to use a rotational analogy (with the string performing rotation like a skipping-rope) and conserve angular momentum. In that case, it will be observed that the decrease in mass of the vibrating portion of the string exactly counteracts the increase in angular velocity caused by the decreased length.

33. Chess Imagine a 4×4 grid with a particle in each cell. These particles move in an L-shape, similar to knights in chess. Every second, a particle moves randomly in one of the cells it can access with an L jump. Two or more particles are allowed to occupy the same cell. After some time, this system will reach an equilibrium. If the (statistical) temperature of the system and thus each cell is $T = 1$ mK, all the cells will have a definite energy level. Find the difference between the highest and lowest energy states in eV.

Solution 33: If we reach a steady state, we can assign a probability $p(x)$ to each cell x. We can think of probability jumping, instead of the ball jumping. So at one step, all the probability in x disappears, and we need some probability jump back into it.

The probability jumping into x is:

$$p(x) = \sum_{u \in \text{neighbours(x)}} p(u)p(u \to x)$$

Now $p(u \to x)$ is just $\frac{1}{\deg(u)}$. The degree of a cell, denoted deg is the number of other cells that can reach it

$$p(x) = \sum_{u \in \text{neighbours(x)}} \frac{p(u)}{\deg(u)}$$

Here, we can see that if the system reaches a steady state, $p(x)$ will be proportional to $\deg(x)$. So the solution is

$$p(x) = \frac{\deg(x)}{\sum_u \deg(u)}$$

Which we can check in practice (with something like Mathematica), and it works.

For our 4×4 grid, the degrees of the cells are:

$$\begin{matrix} 2 & 3 & 3 & 2 \\ 3 & 4 & 4 & 3 \\ 3 & 4 & 4 & 3 \\ 2 & 3 & 3 & 2 \end{matrix}$$

The probabilities will be proportional to these.

Now let's find the energies. Denote the energy of cell i as E_i. Also, we will denote $\beta = \frac{1}{k_B T}$. By the Boltzmann distribution:

$$\frac{p(E_1)}{p(E_2)} = e^{\beta(E_2 - E_1)}$$

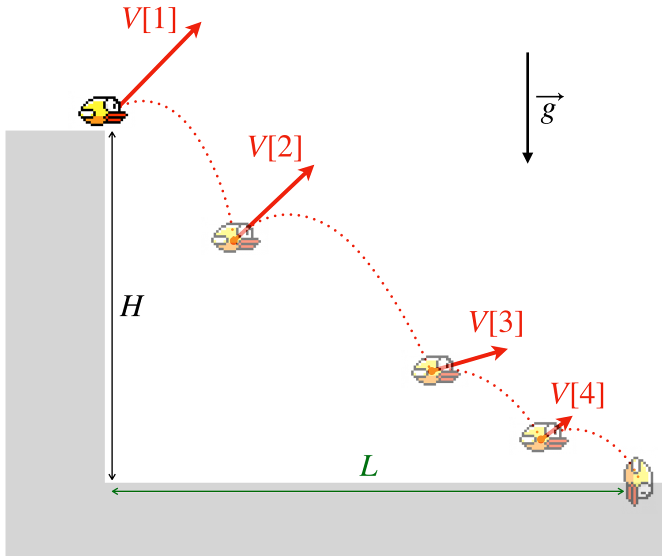
$$E_2 - E_1 = k_B T \log \frac{p(E_1)}{p(E_2)}$$

This means that the difference in energies is maximized for biggest ratio. Which gives:

$$\Delta E_{max} = k_B T \log \frac{4}{2} = \boxed{5.9728 \cdot 10^{-8} \text{eV.}}$$

34. Conducting sheets [This problem has been removed from the test.]

35. A Tired Flappy Bird A flappy bird can jump multiple times in the air. Each time it jumps mid-air, it can suddenly change its speed and direction. For every jump, the bird can decide when to jump and in which direction. Between jumps, the bird falls freely under gravity, which pulls it down at the acceleration g. Say, our tired flappy bird starts off the cliff of height H with the jumping velocity $V[1] = V_0$. Subsequent jumps in mid-air have decreasing velocities, i.e. the n-th jump has speed $V[n] = V_0/n$ ($n > 1$). This majestic Vietnamese animal wants to travel as far as possible horizontally before it lands on the ground. Find the maximum horizontal distance the bird can travel (denoted as L in the figure below) in meters, given that $H = 100\text{m}$ and $V_0 = 10\text{m/s}$. *Note that each jumping velocity is the total speed of the bird after the jump (rather than e.g. adding to its speed before the jump).*



Solution 34: Let us start with the first jump, starting at the location $(x_0, y_0) = (0, H)$. Determining the envelope of a projectile launched with velocity $V[1]$ is a well-known problem, where the bound of all possible trajectories forms a parabola that satisfies the following equation:

$$y_1 = H_{01} - A_{01}x_1^2 \quad , \quad \text{in which} \quad H_{01} = H + \frac{V[1]^2}{2g} \quad \text{and} \quad A_{01} = \frac{g}{2V[1]^2} \quad . \tag{10}$$

If the second jump starts at a position (x_1, y_1) on this envelope, then all possible position the bird can reach (at least, before the third jump) can be described by:

$$y_2 = (y_1 + H_{21}) - A_{21}(x_2 - x_1)^2 \quad , \quad \text{in which} \quad H_{21} = \frac{V[2]^2}{2g} \quad \text{and} \quad A_{21} = \frac{g}{2V[2]^2} \quad . \tag{11}$$

i.e.

$$y_2 = H_{01} + H_{21} - A_{01}x_1^2 - A_{21}(x_2 - x_1)^2.$$

All the points $(x, y) = (x_2, y_2)$ which are reachable within two jumps will have a/multiple respective x_1 for which the above equation is fulfilled. The equation is qudratic with respect to x_1 so there will exist a respective x_1 if the discriminant of the equation (wrt. x_1) is non-negative. In standard form the quadratic equation wrt. x_1 is

$$(A_{01} + A_{21})x_1^2 - 2A_{21}x_2x_1 + (y_2 + A_{21}x_2^2 - H_{01} - H_{21}) = 0,$$

so from a non-negative discriminant we get

$$\begin{aligned} 0 &\geq (A_{01} + A_{21})y_2 + A_{01}A_{21}x_2^2 - (A_{01} + A_{21})(H_{01} + H_{21}) \\ \iff \quad y_2 &\leq (H_{01} + H_{21}) - \left(A_{01}^{-1} + A_{12}^{-1}\right)^{-1} x_2^2 \quad . \end{aligned} \tag{12}$$

The envelope curve for two jumps is then the one for which the above inequality becomes an equality.

We can carry on the procedure inductively for an arbitrary n-th jump and get the envelope curve to be

$$y_n = \left(\sum_{j=1}^{n} H_{j,j-1}\right) - \left(\sum_{j=1}^{n} A_{j,j-1}^{-1}\right)^{-1} x_n^2,$$

where we can carry on the calculation for $n \to \infty$ with the Basel summation series:

$$\sum_{j=1}^{\infty} H_{j,j-1} = H + \sum_{j=1}^{\infty} \frac{V[j]^2}{2g} = H + \frac{V_0^2}{2g} \sum_{j=1}^{\infty} j^{-2} = H + \frac{\pi^2}{12} \frac{V_0^2}{g} \quad ,$$
$$\left(\sum_{j=1}^{\infty} A_{j,j-1}^{-1} \right)^{-1} = \left(\sum_{j=1}^{\infty} \frac{2V[j]^2}{g} \right)^{-1} = \frac{g}{2V_0^2} \left(\sum_{j=1}^{\infty} j^{-2} \right)^{-1} = \frac{3}{\pi^2} \frac{g}{V_0^2} \quad . \tag{13}$$

i.e. the envelope curve for the tired flappy bird is

$$y = H + \frac{\pi^2}{12} \frac{V_0^2}{g} - \frac{3}{\pi^2} \frac{g}{V_0^2} x^2.$$

The furthest (horizontal) distance is the x-coordinate ($x = L > 0$) where the envelope curve intersects the ground $y = 0$:

$$0 = \left(H + \frac{\pi^2}{12} \frac{V_0^2}{g} \right) - \left(\frac{3}{\pi^2} \frac{g}{V_0^2} \right) L^2$$
$$\implies \quad L = \frac{\pi^2}{6} \left(1 + \frac{12}{\pi^2} \frac{gH}{V_0^2} \right)^{1/2} \frac{V_0^2}{g} \approx \boxed{60.32 \text{ m}}. \tag{14}$$

* This puzzle was created with helps from Long T. Nguyen.