2024 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 4 long answer questions over 2 full days from August 30, 12:00AM UTC to September 1, 12:00 AM UTC.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a Google Form where they are allowed to submit up 50 MB of data for each problem solution. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.
- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.
- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^5\ \mathrm{N/m^2} = 1.01 \cdot 10^5\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

1 Penned Particles

A Penning trap is a device used to store charged particles using static magnetic and electric fields. In this problem we will investigate the motion of an ion inside the trap.

1.1

(a) The trap is a cylinder, parallel to the z-axis, with the origin at the center. Inside, the electric potential is $V = V_0 \frac{z^2 - r^2}{2d^2}$, where d is the characteristic dimension of the trap. In order to generate the quadrupole field inside, there are two sets of electrodes: two endcaps and the ring electrode, which are held at potential difference V_0, and are solids of revolution. Refer to part (e) for a diagram. Let the minimum distance between endcaps be $2z_0$, and the smallest inside diameter of the ring be $2r_0$.

- Take a cross section parallel to the z-axis through the origin. What are the equations of the cross section of the ring and endcap electrodes?

- Express d in terms of r_0 and z_0.

(b) The magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ is homogeneous inside the trap. Suppose we have a particle with charge q and mass m. Assume its speed is nonrelativistic, and neglect energy loss from radiation. Throughout the rest of the problem, assume q is positive.

- The z-axis motion is simple harmonic. Find the angular frequency ω_z.

- Write the differential equation for the motion in the xy-plane.

- Suppose $\omega_z = 0$. Solve the differential equation, and find the angular frequency of the motion ω_c. This is the *cyclotron* frequency.

Typically, $\omega_c \gg \omega_z$. Assume this for the rest of the problem.

(c) The motion of the electron in the xy-plane consists of two separate uniform circular motions overlaid on top of each other. One is the cyclotron motion and the other is the *magnetron* motion. Find expressions for the angular frequencies of the cyclotron motion and the magnetron motion, in terms of ω_z and ω_c.

1.2

(d) We will now consider the effects of radiation. Typically, the magnetron motion has a much lower frequency than the cyclotron motion, so the decay of the magnetron motion is negligible. The power radiated by an accelerating particle is:

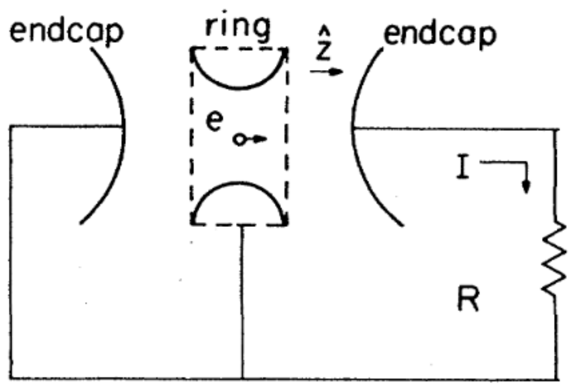
$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}.$$

- The energy of the orbit decays as e^{-t/γ_c}. Find γ_c.

- Now consider the radiation damping of the axial motion. The energy of the oscillation decays as e^{-t/γ_z}. Find γ_z.

(e) For an electron at typical ω_c, γ_c is quite small, allowing for easy damping. However, γ_z is much larger, and for a proton, radiation damping is insignificant. In order to cool large particles, a circuit is used instead. We first consider axial damping.

The oscillations of the ion induce image charges in the electrode, which can be interpreted as a current I. See the following circuit:



You may ignore the quadrupole potential in this part.

- There will be a potential difference of IR between the endcap and the ring (as well as the other endcap). This will produce an electric field $E\hat{\mathbf{z}}$ proportional to I inside the trap. Find E, up to a constant factor κ, which depends on the geometry of the electrodes. *Hint*: if the endcaps are infinite flat planes, κ is equal to 1.

- Consider the power lost through the resistor. Use this to derive the force on the ion, $f = -m\zeta\dot{z}$. Write an expression for ζ.

(f) To conclude, we will consider how to cool the magnetron motion (decrease its radius).

- Find the total energy of the magnetron motion. Assume $z = 0$.

The process works as follows. We shine photons of energy $\hbar(\omega_z + \omega_m)$, which interact with the ion. Let the quantum numbers of the z motion and the magnetron motion be k and l respectively. Then, the cooling transition is from $(k, l) \to (k+1, l-1)$, and the heating transition is from $(k, l) \to (k-1, l+1)$. Using quantum mechanics, we can derive that these happen at rates proportional to $(k+1)l$ and $k(l+1)$ respectively. The magnetron motion will be cooled until $l = k$, at which point it will be in equilibrium, and there is no long term change in temperature.

- We will now derive the equilibrium energy of the magnetron motion. Assume that at equilibrium, the axial and magnetron motions are at temperatures T_z and T_m respectively. As we continue to shine photons, consider the change in entropy. Use this to derive T_m in terms of ω_m, ω_z, and T_z.

Penned Particles – Solution

Total: **35.0 pts**

The problem originally stated the potential as $V = V_0 \frac{z^2 - r^2}{2d^2}$. However, this has a non-zero Laplacian, so it is invalid; a valid potential would be $V = V_0 \frac{2z^2 - r^2}{2d^2}$. We will give full points for using either potential throughout the problem.

(a) The electrodes are equipotentials. The potential of the endcaps is $V = V_0 \frac{z_0^2}{2d^2}$, while the potential of the ring is $V_0 \frac{-r_0^2}{2d^2}$. Thus, $d^2 = \frac{1}{2}(z_0^2 + r_0^2)$. The equation for the endcaps is $z^2 = z_0^2 + r^2$, and for the ring is $z^2 = r^2 - r_0^2$.

Grading Scheme: **5.0 pts** total.

- 1.5 points for realizing the electrodes are equipotentials, or equivalent.
- 1.5 points for correct equations for the cross sections: $z^2 = z_0^2 + r^2$ and $z^2 = r^2 - r_0^2$, OR

  $z^2 = z_0^2 + \frac{1}{2}r^2$ and $z^2 = \frac{1}{2}r^2 - \frac{1}{2}r_0^2$.
- 2.0 points for calculation of potential of endcaps and and ring, and using this to find an expression for d: $d = \frac{1}{2}(z_0^2 + r_0^2)$, OR $d = \frac{1}{2}(2z_0^2 + r_0^2)$.

(b) Because $E = -V_0 \frac{z}{d^2}$, we find $\omega_z = \sqrt{\frac{qV_0}{md^2}}$. Then, the equations of motion are:

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

$$\ddot{\mathbf{r}} = q(V_0\mathbf{r}/md^2 + \dot{\mathbf{r}} \times \mathbf{B}/m)$$

$$\ddot{\mathbf{r}} = \omega_z^2 \mathbf{r} + q\dot{\mathbf{r}} \times \mathbf{B}/m$$

Some contestants wrote their equations in terms of x and y coordinates; we will also accept those as valid.

We have $\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}/m$. It's evident the particle performs uniform circular motion at frequency $\omega_c = \frac{qB_0}{m}$. Note that ω_c points opposite B.

Grading Scheme: **7.0 pts** total.

- 2 points for correct axial angular frequency: $\omega_z = \sqrt{\frac{qV_0}{md^2}}$ OR $\omega_z = \sqrt{\frac{2qV_0}{md^2}}$.
- 2 points for correct differential equations.
- 3 points for the correct cyclotron frequency.

(c) To start, we have $\ddot{\mathbf{r}} = \omega_z^2 \mathbf{r} + \dot{\mathbf{r}} \times \omega_c$. Suppose we have $\mathbf{r} = (r, \omega t)$ in polar coordinates. Then $\dot{\mathbf{r}} = \omega r \hat{\theta}$, and $\ddot{\mathbf{r}} = -\omega^2 \mathbf{r}$. We plug these into the differential equation, to get

$$-\omega^2 \mathbf{r} = \omega_z^2 \mathbf{r} - \omega\omega_c \mathbf{r}.$$

Using the quadratic equation, this gives us

$$\omega = \frac{\omega_c \pm \sqrt{\omega_c^2 - 4\omega_z^2}}{2}.$$

Recall that when $\omega_z = 0$, the cyclotron motion has frequency ω_c, so we take

$$\omega_c' = \frac{w_c + \sqrt{w_c^2 - 4\omega_z^2}}{2}.$$

This is less than ω_c because of the radial potential pushing the ion outwards. Then, the magnetron frequency is

$$\omega_m = \frac{\omega_c - \sqrt{\omega_c^2 - 4\omega_z^2}}{2} \approx \frac{\omega_z^2}{\omega_c}.$$

We thus have $\omega_c \gg \omega_z \gg \omega_m$.

Grading Scheme: **8.0 pts** total.

- 3 points for guessing a circular/periodic/complex exponential solution to the differential equation.
- 4 points for solving the differential equation and correctly deriving the two possible frequencies.
- 1 point for correctly identifying which frequency is the magnetron frequency and which is the cyclotron frequency.

(d) The original formula for power provided was missing a factor of π; we will not penalize solutions that are missing this factor. Note that $E = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m\omega_c^2 r^2$. The acceleration is $\omega_c^2 r$, so

$$P = \frac{dE}{dt} = -\frac{q^2\omega_c^4 r^2}{6\varepsilon_0 c^3} = -E\frac{q^2\omega_c^2}{3m\varepsilon_0 c^3}.$$

Thus,

$$\gamma_c = \frac{3m\varepsilon_0 c^3}{q^2\omega_c^2}.$$

The energy is $qV_0\frac{z_m^2}{2d^2} = \frac{1}{2}m\omega_z^2 z_m^2$, where z_m is the amplitude of the oscillation. We have $z = z_m \sin\omega_z t$, and $a = -\omega_z^2 z$. Averaged over one oscillation, $\langle a^2\rangle = \frac{1}{2}\omega_z^4 z_m^2$. We then have

$$P = -\frac{q^2 \cdot \frac{1}{2}\omega_z^4 z_m^2}{6\varepsilon_0 c^3}.$$

We then have

$$\gamma_z = \frac{6m\varepsilon_0 c^3}{q^2\omega_z^2}.$$

Grading Scheme: **5.0 pts** total.

- 0.5 points for correct expression for the energy of the cyclotron motion.
- 1 point for correct expression for the power.
- 0.5 points for correct damping constant γ_c.
- 1 point for correct expression for the energy of the axial motion.
- 1.5 points for the correct expression for the time-averaged power.
- 0.5 points for correct damping constant γ_z.

(e) Supposing that the endcaps were infinite flat planes, the electric field will be $\frac{IR}{2z_0}$. So we have $E = \frac{\kappa IR}{2z_0}$.

The power on the ion is $F\dot{z} = qE\dot{z}$, which we equate to RI^2:

$$q\frac{\kappa IR}{2z_0}\dot{z} = RI^2 \Rightarrow$$

$$I = \frac{\kappa q\dot{z}}{2z_0}.$$

Then

$$F = q\frac{\kappa IR}{2z_0} = q^2\frac{\kappa^2 R}{4z_0^2}\dot{z}.$$

So we have

$$\zeta = \frac{R}{m}\left(\frac{\kappa q}{2z_0}\right)^2.$$

Grading Scheme: **5.0 pts** total.

- 1.5 points for the correct electric field at the center of the trap.
- 1.5 points for correct power on the ion, and equating it to the power lost through the resistor.
- 1.5 points for solving for current in terms of $\dot{z}$, or equivalent derivation of the force in terms of $\dot{z}$.
- 0.5 point for correct expression for ζ.

(f) Note that:

$$E = \frac{1}{2}m\omega_m^2 r^2 + qV = \frac{1}{2}m\omega_m^2 r^2 - qV_0 r^2/2d^2 =$$

$$\frac{1}{2}m\omega_m^2 r^2 - \frac{1}{2}m\omega_z^2 r^2.$$

Since $\omega_z \gg \omega_m$ this is obviously negative.

Since the system is in equilibrium, over the long run the change in entropy must be zero. When photons interact with the ion, we can view it as adding heat $Q_z = \hbar\omega_z$ and $Q_m = \hbar\omega_m$ to reservoirs at temperatures T_z and T_m respectively. The change in entropy is

$$\Delta S = \frac{Q_z}{T_z} + \frac{Q_m}{T_m} = 0.$$

. So, we have

$$T_m = -\frac{\omega_m}{\omega_z}T_z.$$

Many teams reported T_m to be positive. Here is a subtle point about negative temperature. Earlier in the problem, we derived that the total energy of the magnetron motion was negative. If we were to add energy, the radius would shrink. Very informally, temperature is defined such that $TdS = dE$, and intuitively, an increase in the radius of the orbit is an increase in entropy. (This can all be formalized with quantum mechanics). Adding energy decreases the entropy, which is why T_m is negative.

Grading Scheme: **5.0 pts** total.

- 1 point for correct expression for the energy (up to first order).
- 3 points for conserving entropy.
- 1 point for correct expression for T_m. -0.5 points if positive.

2 Bouncy Bubble

In this problem, we will investigate the interaction between fast oscillations and gradual changes in a physical system.

2.1

A large volume of incompressible, non-viscous liquid with density ρ is kept at temperature T_c and pressure P_c. A spherical bubble consisting of N particles of ideal gas with temperature $T_0 > T_c$ is introduced into the liquid. Neglect surface tension and any heat transfer between the liquid and the gas.

(a) Find the equilibrium radius R_0 of the bubble.

The bubble's radius is perturbed slightly from equilibrium and its oscillations are observed; the gas remains near thermal equilibrium at all times. Assume that the motion of the liquid is laminar and radial, and that the density of the gas is negligible compared to ρ. You may express future answers in terms of R_0.

(b) Find the frequency ω of the bubble's small oscillations.

2.2

Now, assume that the interface between the gas and the liquid has thermal conductance per unit area κ. Then, because of heat loss, the bubble will shrink over time, approaching a final radius R_f (which you may use in future answers). The shrinkage is slow enough that the kinetic energy of the liquid can be neglected.

(c) If the bubble starts at radius R_0, find the approximate time τ until it shrinks to radius $(R_0 + R_f)/2$. Express your answer to the lowest order in the quantity $\alpha = R_0/R_f - 1$.

(d) Next, the bubble starts off oscillating around R_0 with amplitude $R_0\delta_0$, where $\delta_0 \ll 1$; assume that the oscillations are much faster than the shrinkage. Find the time-averaged final radius R'_f of the bubble, to the lowest order in δ_0. Qualitatively explain the reason for any difference between R'_f and R_f.

(e) Given the situation in part (d), find the approximate time τ' until the bubble's time-averaged radius shrinks to $(R_0 + R'_f)/2$, to the lowest orders in $\alpha' = R_0/R'_f - 1$ and δ_0.

Bouncy Bubble – Solution

Total: **25.0 pts**

(a) At equilibrium, the pressure in the bubble is P_c. Using the Ideal Gas Law:

$$P_c V = P_c \cdot \frac{4}{3}\pi R_0^3 = NkT_0 \Rightarrow \boxed{R_0 = \sqrt[3]{\frac{3NkT_0}{4\pi P_c}}}$$

Grading Scheme: **1.0 pts** total.

- 0.5 pts for use of the Ideal Gas Law.
- 0.5 pts for a correct expression for R_0.

(b) Let the bubble have radius $R_0 + x$. Setting $U = 0$ at equilibrium, the work-energy theorem implies:

$$U = \int_0^x 4\pi R_0^2 (P_c - P)\, da$$

Using $PV^{5/3} = \text{const}$:

$$\begin{aligned} U &= \int_0^x 4\pi R_0^2 \left(P_c - \frac{P_c \left(\frac{4}{3}\pi R_0^3\right)^{5/3}}{\left(\frac{4}{3}\pi (R_0+a)^3\right)^{5/3}} \right) da \\ &= \int_0^x 4\pi R_0^2 P_c \left(1 - \left(1 + \frac{a}{R_0}\right)^{-5} \right) da \\ &\approx \int_0^x 20\pi P_c R_0 a\, da \\ &= 10\pi P_c R_0 x^2 \end{aligned}$$

The kinetic energy comes from the motion of the liquid. If the bubble's radius is changing at rate $\dot{x}$, the liquid at radius r must have speed $\frac{R_0^2}{r^2}\dot{x}$ by incompressibility. Thus:

$$K = \int_R^\infty \frac{1}{2}(4\pi\rho r^2\, dr)\left(\frac{R_0^2}{r^2}\dot{x}\right)^2 = 2\pi\rho R_0^4 \dot{x}^2 \int_R^\infty \frac{1}{r^2}\, dr = 2\pi\rho R_0^3 \dot{x}^2$$

We find $\boxed{\omega = \frac{1}{R_0}\sqrt{\frac{5P_c}{\rho}}}$.

Grading Scheme: **5.0 pts** total

- 1.0 pts for using the work-energy theorem to find an integral for U in terms of $P_c - P$
- 1.0 pts for using the adiabatic condition to linearize $P_c - P$.
- 0.5 pts for a correct expression for U.
- 1.0 pts for using incompressibility to derive the velocity profile of the liquid, $v(r) \propto 1/r^2$.
- 0.5 pts for a correct expression for K.
- 1.0 pts for a correct expression for ω.

(c) Because the bubble begins at equilibrium and shrinks slowly, it will remain near equilibrium. Thus, we will always have $T = \frac{4\pi P_c r^3}{3Nk}$. Because the external pressure is constant:

$$\dot{H} = \dot{Q} = -4\pi r^2 \kappa (T - T_c)$$

Using $H = \frac{5}{2}NkT$, we have:

$$\dot{H} = \frac{5}{2}Nk\dot{T} = \frac{5}{2}Nk\frac{dT}{dr}\dot{r} = 10\pi P_c r^2 \dot{r} \Rightarrow \dot{r} = -\frac{2\kappa}{5P_c}(T - T_c) = -\frac{8\pi\kappa}{15Nk}\left(r^3 - \frac{3NkT_c}{4\pi P_c}\right)$$

Note that $R_f^3 = \frac{3NkT_c}{4\pi P_c}$. Separating variables:

$$\int_{R_0}^{(R_0+R_f)/2} \frac{1}{r^3 - R_f^3}\, dr = \int_0^\tau -\frac{8\pi\kappa}{15Nk}\, dt \Rightarrow \frac{1}{R_f^2}\int_{1+\alpha/2}^{1+\alpha} \frac{1}{x^3-1}\, dx = \frac{8\pi\kappa}{15Nk}\tau$$

$$\int_{1+\alpha/2}^{1+\alpha} \frac{1}{x^3-1}\, dx \approx \int_{1+\alpha/2}^{1+\alpha} \frac{1}{3(x-1)} - \frac{1}{3}\, dx = \frac{\ln(2)}{3} - \frac{\alpha}{6}$$

Thus, $\boxed{\tau = \frac{5\ln(2)Nk}{8\pi\kappa R_f^2}\left(1 - \frac{\alpha}{2\ln(2)}\right)}$.

Grading Scheme: **5.0 pts** total.

- 0.5 pts for a correct expression for $\dot{Q}$ in terms of $T - T_c$
- 1.0 pts for equating $\dot{Q}$ to $\dot{H}$ or $C_p\dot{T}$ due to constant external pressure.
- 1.5 pts for substituting the expression for T and obtaining a correct differential equation in r.
- 1.0 pts for separating variables and approximating the integrand as shown.
- 1.0 pts for a correct final answer, or 0.5 pts for an answer with a correct leading term.

(d) Let the oscillating radius be $r(1 + \delta\cos(\omega t))$. We claim that the total energy E of the oscillations is proportional to ω, giving $\delta = \frac{R_0^2}{r^2}\delta_0$. To see this, we consider the *adiabatic invariant*, I: the area enclosed by the system's motion in phase space. For any simple harmonic oscillator, $I = 2\pi E/\omega$. Crucially, I remains approximately constant under slow external changes, giving $E \propto \omega$ as desired. You can read more here.

Using $TV^{2/3} = \text{const}$, the oscillating temperature is $\frac{T}{(1+\delta\cos(\omega t))^2}$. Thus, we have:

$$\begin{aligned}
\dot{Q} &= -4\pi\kappa r^2 (1+\delta\cos(\omega t))^2 \left(\frac{T}{(1+\delta\cos(\omega t))^2} - T_c\right) \\
&= -4\pi\kappa r^2 \left(T - (1 + 2\delta\cos(\omega t) + \delta^2\cos^2(\omega t))T_c\right) \\
\Rightarrow \left\langle \dot{Q} \right\rangle &= -4\pi\kappa r^2 \left(T - \left(1 + \frac{\delta^2}{2}\right)T_c\right) \\
&= -4\pi\kappa r^2 T_c \left(\frac{r^3}{R_f^3} - 1 - \frac{R_0^4\delta_0^2}{2r^4}\right)
\end{aligned}$$

At equilibrium, we must have $\left\langle \dot{Q} \right\rangle = 0$. Thus, assuming that $R_f' = R_f(1+c)$:

$$(1+c)^3 - 1 - \frac{R_0^4\delta_0^2}{2R_f^4(1+c)^4} \approx 3c - \frac{R_0^4\delta_0^2}{2R_f^4} = 0 \Rightarrow \boxed{R_f' = R_f\left(1 + \frac{R_0^4\delta_0^2}{6R_f^4}\right)}$$

The fundamental reason why $R_f' > R_f$ is that the rate of heat loss is a concave function under adiabatic perturbations of r. Thus, the rate of heat loss under the fast, roughly adiabatic oscillations averages out to less than it does with no oscillation, so the equilibrium radius is slightly larger.

Grading Scheme: **9.0 pts** total.

- 3.0 pts for use of the adiabatic invariant or similar to derive $\delta(r)$.
- 1.0 pts for obtaining $\delta(r) \propto 1/r^2$.
- 1.0 pts for finding an expression for the oscillating temperature using the adiabatic condition.
- 1.5 pts for substituting the oscillating radius and temperature into $\dot{Q}$ and time-averaging
- 0.5 pts for setting $\langle \dot{Q} \rangle = 0$ at equilibrium.
- 1.0 pts for a correct final answer.
- 1.0 pts for mentioning the concavity of $\dot{Q}$ as the cause of the discrepancy between R_f and R_f', or 0.5 points for only mentioning nonlinearity.

(e) Now, we must consider the change in energy of the oscillations in addition to the change in H. Thus:

$$\dot{H} + \dot{E} = \dot{H} + \frac{dE}{dr}\dot{r} = \left\langle \dot{Q} \right\rangle$$

Note that we do not time-average $\dot{H}$ because it is a state variable (when the system crosses through equilibrium, H is uniquely determined by r). Then, using $E = 10\pi P_c r^3 \delta^2 = 10\pi P_c \frac{R_0^4 \delta_0^2}{r}$:

$$10\pi P_c r^2 \dot{r} - 10\pi P_c \frac{R_0^4 \delta_0^2}{r^2}\dot{r} = -4\pi\kappa r^2 T_c \left(\frac{r^3}{R_f^3} - 1 - \frac{R_0^4 \delta_0^2}{2r^4} \right)$$

$$\Rightarrow \dot{r} = -\frac{8\pi\kappa}{15Nk}\left(r^3 - R_f^3\left(1 + \frac{R_0^4\delta_0^2}{2r^4}\right)\right) \Big/ \left(1 - \frac{R_0^4\delta_0^2}{r^4}\right) \approx -\frac{8\pi\kappa}{15Nk}\frac{r^3 - R_f'^3}{1-\delta_0^2}$$

Here, we take $\frac{R_0^4}{r^4} \approx 1$ because we don't care about the cross term between δ_0 and α'. Thus, we have:

$$\tau' = \frac{15Nk(1-\delta_0^2)}{8\pi\kappa}\int_{R_0}^{(R_0+R_f')/2} \frac{1}{r^3 - R_f'^3}\,dr \approx \frac{5\ln(2)Nk}{8\pi\kappa R_f'^2}\left(1 - \frac{\alpha'}{2\ln(2)}\right)(1-\delta_0^2)$$

Using $R_f' \approx R_f\left(1 + \frac{\delta_0^2}{6}\right)$, we find $\boxed{\tau' = \frac{5\ln(2)Nk}{8\pi\kappa R_f^2}\left(1 - \frac{\alpha'}{2\ln(2)}\right)\left(1 - \frac{4\delta_0^2}{3}\right)}$.

Grading Scheme: **5.0 pts** total.

- 1.0 pts for substituting the time-averaged $\dot{Q}$ into the differential equation; 0.5 pts if time-averaging $\dot{H}$ or $C_p\dot{T}$ as well.
- 2.0 pts for realizing that the energy of the oscillations changes and adding this term into the differential equation.
- 1.0 pts for obtaining a differential equation analogous to the equation given for part (c), with the substitution of R_f' for R_f, and using the same techniques to obtain τ'.
- 1.0 pts for a correct final answer, or 0.5 pts for a final answer with the right correction for δ_0.

3 Stellar Shaping

In this problem, we investigate the formation of stellar systems.

3.1

Consider a cloud of dust of radius R of mass M with particles of mass m, all held at a constant temperature T. Assume that $kT \gg GMm/R$; i.e. the particles are far enough apart such that gravitational interactions are nearly negligible.

(a) What is the expected value and variance of the angular momentum of one particle in the $\hat{x}$ direction?

(b) What is variance in the total angular momentum of the cloud, $\langle L^2 \rangle$?

3.2

Suppose some density fluctuations occur, which leads this cloud of gas into gravitational collapse. Now, we must take gravitational interaction into account; assume that the cloud remains at thermal equilibrium and that the total energy of the cloud remains constant–the work done by the exterior gas is small.

(c) Assume that the cloud remains spherically symmetric. Find the approximate distribution of densities $\rho(r)$. You can use the new equilibrium temperature in your expression, which will be calculated in part (f). The model you find should work under the limit

$$r^2 \gg k_b T / Gm.$$

(d) What is the new radius of the cloud, R'?

(e) Find the angular velocity ω of the cloud, assuming that the cloud rotates uniformly. Take the total angular momentum of the cloud to be $\sqrt{\langle L^2 \rangle}\,\hat{\mathbf{z}}$, which you found in part (b).

(f) What is the new temperature of the cloud, T'?

3.3

The nebula is not at its most stable state because of the high angular velocity. Suppose that the part of the cloud that reaches beyond a critical density limit ρ_c collapses and begins forming a star.

(g) Find the initial radius of collapse, R_c, and the mass of the star M_s. Assume the radius of the star is a lot smaller than R_c.

For the last two parts, we will assume that the gravitational potential is quadratic, $U = \frac{1}{2}k(x^2 + y^2 + z^2)$, and the angular velocity of the particles is ω. Leave answers in terms of the variables given in this part.

(h) Suppose all the leftover material, some N particles at temperature T', begins to settle into a gas. What is the expected value for r^2, the distance of these particles to the axis of rotation, once they reach their most stable state?

(i) What is the approximate variance in the orbital inclination for this leftover material–that eventually begins to form asteroids and planets? (to first order in ω^2)

Stellar Shaping – Solution

Total: **35.0 pts**

(a) The expected value for the angular momentum in one dimension is 0 by symmetry.

$$l_x^2 = m^2(yv_z - zv_y)^2 = m^2(y^2v_z^2 - 2yzv_zv_y + z^2v_y^2)$$

Knowing that position and velocity are independent quantities, we take the expected value of this equation, obtaining

$$\langle l_x^2 \rangle = m^2(\langle y^2 \rangle \langle v_z^2 \rangle - 2\langle yzv_zv_y \rangle + \langle z^2 \rangle \langle v_y^2 \rangle) = m^2(2\langle x^2 \rangle \langle v_x^2 \rangle)$$

where we used the fact that the dimensions are symmetric, so the first and third term are equal and the second term is zero by symmetry. Now,

$$\langle x^2 \rangle = \langle r^2 \rangle /3 = \frac{\int_0^R r^2 \cdot 4\pi r^2 \, dr}{4\pi R^3} = \frac{1}{5}R^2$$

By the equipartition theorem and/or using the Boltzmann Distribution,

$$\langle v_x^2 \rangle = k_bT/m.$$

So then, the variance is $m^2(2(\frac{1}{5}R^2)(k_bT/m)) = \boxed{\frac{2}{5}mkTR^2}$.

There are many ways to solve this–averaging angles, using cylindrical shells, finding the total angular momentum and using symmetry–all of these return the same answer and are equally valid.

Grading Scheme: **4.0 pts** total.

- 0.5 pts for recognizing the expected value is 0 by symmetry.
- 0.5 pts for using the equipartition theorem in some way.
- 0.5 pts for realizing position and velocity are independent.
- 2.5 pts for getting the correct expression for the variance.

(b) In total, there are M/m particles. Then, by the central limit theorem, the total angular momentum of the cloud has a Gaussian distribution with variance in one dimension

$$\langle L_x^2 \rangle = \frac{M}{m} \langle l_x^2 \rangle = \frac{M}{m}\frac{2}{5}mkTR^2.$$

Then

$$\langle L^2 \rangle = 3\langle L_x^2 \rangle = \boxed{\frac{6}{5}Mk_bTR^2}.$$

Grading Scheme: **2.0 pts** total.

- 0.5 pts for using some form of the central limit theorem.
- 0.5 pts for using symmetry to find L^2. 4
- 1 pt for correct answer.

(c) Let T' be the new temperature of the cloud after it reaches thermal equilibrium, which will be calculated in part(f). We can use hydro-static equilibrium:

$$\frac{dP}{dr} = -\rho(r)g = -\rho(r)\frac{GM(r)}{r^2}$$

with

$$P(r) = \rho(r)kT'/m$$

and

$$M(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$$

This reduces to

$$\frac{d}{dr}(\frac{r^2}{\rho}\frac{d\rho}{dr}) = -\frac{4\pi Gm}{k_b T'}r^2\rho.$$

Since there are only powers in r in this differential equation, we infer a power-law relationship, $\rho = ar^n$. Plugging this back, we get

$$\begin{aligned}\frac{d}{dr}(\frac{r^2}{ar^n}\frac{d(ar^n)}{dr}) &= -\frac{4\pi Gm}{k_b T'}r^2(ar^n)\\ \frac{d}{dr}(nr) &= -\frac{4\pi Gm}{k_b T'}ar^{n+2}\\ n &= -\frac{4\pi Gm}{k_b T'}ar^{n+2}.\end{aligned}$$

This results in a solution only when $n = -2$, which makes $a = \frac{k_b T}{2\pi Gm}$. So, then our distribution of densities should be

$$\rho(r) = \boxed{\frac{k_b T'}{2\pi Gmr^2}}.$$

However, this solution is nonphysical. For a spherically symmetric distribution we expect the density to be finite and that $\frac{d\rho}{dr} = 0$ at the center. However, if we consider a small perturbation to our initial function,

$$\rho(r) \approx \frac{k_b T'}{2\pi Gm(r^2 + \delta^2)}$$

both these conditions are satisfied. However, this modified function now doesn't solve our differential equations. But in the limit

$$r^2 \gg \delta^2$$

this is true, thus for all subsequent calculations we can use our aforementioned solution.

In differential equations lingo, this is called the singular solution to the differential equation. Normally when solving a second-order differential equation, we would have 2 parameters to vary in order to match initial conditions. However, all solutions eventually approach this in the limit of r. The other solutions to these system of differential equations are non-analytic; see Emden–Chandrasekhar Equation for more information regarding this. Below is a numerical solution I performed in Python to solve the differential equation.

The solutions to these equations are used to model the core of a star, as only the isothermal assumption remains valid in that region of the star. Beyond this region, that assumption must be broken as otherwise, there would be infinite mass, requiring us to bound the mass of the star in part (d).

Grading Scheme: **7.0 pts** total.

- 2 pts for using Boltzmann Distribution, Hydro-static Equilibrium, Poission Equations or any physical equivalent.
- 2 pts for correct differential equation (in any equivalent form).

- 1 pt for guessing a power-law solution.
- 1 pt mentioning any physical breakdowns (boundary conditions at $r = 0$) and/or the significance of the limit.
- 1 pt for correct answer.

(d) As aforementioned, we have to use the radius to bound the total mass M of the cloud. Integrating our expression for density, we get

$$M(r) = \frac{2k_bT'}{Gm}r.$$

Then $M(R') = M$ so

$$R' = \boxed{\frac{GMm}{2k_bT'}}.$$

Grading Scheme: **3.0 pts** total.

- 1 pt for recognizing that the radius must be chosen to bound the total mass M.
- 1 pt for integrating the density distribution to find $M(r)$.
- 1 pt for correct answer.

(e) Let's first find the moment of inertia of the ball. Using the new mass distribution, we integrate along spherical shells that have moment of inertia $\frac{2}{3}r^2\,dM$.

$$I = \int_0^M \frac{2}{3}r^2 dM = \int_0^{R'} \frac{2}{3}r^2\left(\frac{M}{R'}\,dr\right) = \frac{2}{9}MR'^2 = \frac{G^2M^3m^2}{18k_b^2T'^2}$$

Then the angular velocity of the cloud would be

$$\omega = L/I = \boxed{\frac{\sqrt{\frac{6}{5}Mk_bTR^2}}{\frac{2}{9}MR'^2}}.$$

The answer can be simplified further but points will be awarded to this step.

Grading Scheme: **4.0 pts** total.

- 1 pt for using integration to find the moment of inertia.
- 1 pt for using conservation of angular momentum.
- 2 pt for correct answer. If initial angular momentum is incorrect, -0.5.

(f) First, let's find the gravitational potential energy. In part (a) we assume that before it collapses it is negligible. Now, however, it is not. We calculate the GPE by imagining the work required to move concentric shells of radius r and mass dM.

$$E_g = -\int_0^M \frac{GM(r)}{r}\,dM = -GM^2/R' = -2\frac{M}{m}k_bT' = -2Nk_bT'$$

$$\begin{aligned}\frac{3}{2}Nk_bT &= \frac{3}{2}Nk_bT' - 2Nk_bT' + \frac{L^2}{2I} \\ &= -\frac{1}{2}Nk_bT' + \frac{L^2(9k_b^2T'^2)}{G^2M^3m^2}\end{aligned}$$

This is a quadratic in terms of T', which can be solved using the quadratic formula.

$$T' = \frac{G^2M^3m^2Nk_b}{36k_b^2L^2}\left(T + \sqrt{T^2 + \frac{216k_bL^2}{G^2M^3m^2N}}\right)$$

We take the correct sign for the root looking at the limiting case when the rotational kinetic energy is negligble compared to the gravitational potential.

Grading Scheme: **5.0 pts** total.

- 1 pt for using integration to find the GPE
- 1 pt for using conservation of energy.
- 1 pt for correct algebraic equation.
- 2 pt for correct answer. If initial angular momentum is incorrect, -0.5.

(g) We find the critical radius R_c of the cloud by solving for R_c in the density distribution.

$$\rho_c = \frac{k_bT'}{2\pi GmR_c^2}$$

$$R_c = \boxed{\sqrt{\frac{k_bT'}{2\pi G\rho_c}}}.$$

$$M_c = MR_c/R' = M\sqrt{\frac{k_bT'}{2\pi Gm\rho_c}} \cdot \frac{2k_bT'}{Gm}$$

Grading Scheme: **1.0 pt** total.

- 0.5 for correct setup.
- 0.5 for correct calculation for mass.

(h) The potential can be written as

$$-\frac{1}{2}m\omega^2r^2 + \frac{1}{2}k(r^2+z^2)$$

where r is the distance to the axis of symmetry. We can write the Boltzmann distribution

$$e^{(\frac{1}{2}m\omega^2-\frac{1}{2}k)r^2}e^{-\frac{1}{2}kz^2}.$$

We recognize from the energy equation that the equipotnential surfaces are squished spheres/elipsoids with varying levels of r with differential volume $\propto r^2\,dr$. This is because the differential volumes are taken between two squished spheres as the radius is the parameter–meaning that the thickness of the differential volume is different based on the height–so it isn't just proportional to the surface area of an elipsoid. We also notice that $\langle r^2\rangle$ for these surfaces should be the same as if they were not squished–full spheres, because of this reason. Thus $\langle r^2\rangle = \frac{2}{3}r^2$. Now we can integrate on these surfaces these to get expected value of r^2 of the entire gas. I'll let $\alpha = \frac{1}{2}k - \frac{1}{2}m\omega^2$ and $\beta = \frac{1}{k_bT}$ for written simplicity.

$$\langle r^2\rangle = \frac{2}{3}\int_0^\infty r^4e^{-\beta\alpha r^2}\,dr \bigg/ \int_0^\infty r^2e^{-\beta\alpha r^2}\,dr = \frac{2}{3}(\frac{3}{8\alpha^2\beta^2})/(\frac{1}{4\alpha\beta}) = \frac{1}{\alpha\beta} = \frac{2k_bT}{k-m\omega^2}$$

We see that this makes sense by considering the case where ω^2 tends to k/m, The expected value for r^2 tends to infinity.

As this part was written somewhat hastily, the second part to this was not added. The particles initially start off with some angular velocity and should stabilize to conserve angular momentum. Here, it would be $L = M\frac{2}{9}R'^2\omega_0 = M\omega\frac{2k_bT}{k-m\omega^2}$ and ω would be found from there as the solution to a quadratic, assuming the star's radius and therefore mass is small. From there, $\langle r^2 \rangle$ could be calculated by plugging back ω in, and finding the new temperature by conserving energy, similarly to the second part. However, as this was not fully clarified, points were awarded for this solution.

Because it wasn't specified whether the potential was per unit mass or not, points were also given to solutions that treated U as per unit mass.

Grading Scheme: **4.0 pt** total.

- 2 pt for finding the correct boltzman distribution.
- 1 pt for using symmetry/equipotential surfaces.
- 1 pt for correct answer.

(i) This equation asks to find the variance in the inclination. First, by symmetry, we know that the expected value of the inclination is 0. Let's call the ratio $\epsilon = b/a = \sqrt{1 - \frac{m}{k}\omega^2}$ of the major and minor axis of the eliptical equipotential surfaces. We know choosing a specific r value doesn't change the expected value for the variance of the inclination, as it just scales the elipsoid up and down. Thus, we can focus on the fraction of volume a certain $d\theta$ takes up with respect to the entire volume. Suppose we take the ellipsoid to have semi-major axis $a = 1$ without loss of generality. At an inclination of θ, we have

$$r^2(\cos^2\theta + \sin^2\theta/\epsilon^2) = 1$$

from the equation of an ellipse.

Then the fraction of volume taken up by $d\theta$ is a cone with volume

$$dV \propto r^3 \cos(\theta)\, d\theta$$

Then the variance in θ is

$$\left\langle \theta^2 \right\rangle(\epsilon) = \int_{-\pi/2}^{\pi/2} \theta^2 r^3 \cos(\theta)\, d\theta \Bigg/ \int_{-\pi/2}^{\pi/2} r^3 \cos(\theta)\, d\theta = \frac{1}{2\epsilon}\int_{-\pi/2}^{\pi/2} \theta^2 r^3 \cos(\theta)\, d\theta$$

The first integral isn't analytically solvable, so we look for an approximation around $\epsilon = 1$. Evaluating the integral yields the variance of a $\cos(\theta)$ distribution, which is known. This is equal to $\frac{\pi^2-8}{4}$. Taking the derivative at $\epsilon = 1$ is equal to

$$\left\langle \theta^2 \right\rangle'(1) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \theta^2 3r^2 (\frac{dr}{d\epsilon}) \cos(\theta)\, d\theta - \frac{\pi^2-8}{4\epsilon^2}\bigg|_{\epsilon=1} = \frac{3}{2}\int_{-\pi/2}^{\pi/2} \theta^2 \sin^2(\theta)\cos(\theta)\, d\theta - \frac{\pi^2-8}{4}$$

using

$$\frac{dr}{d\epsilon}(1) = \frac{2\epsilon(\epsilon^2\cos^2\theta + \sin^2\theta) - \epsilon^2(2\epsilon\cos^2\theta)}{2r(\epsilon^2\cos^2\theta + \sin^2\theta)} = \frac{2 - 2\cos^2(\theta)}{2} = \sin^2(\theta).$$

Evaluating the integral leads to

$$\frac{\pi^2}{4} - \frac{14}{9} - \frac{\pi^2-8}{4} = 2 - \frac{9}{14} = \frac{5}{14}$$

this means we can write the variance to first order as:

$$\langle\theta^2\rangle \approx \frac{\pi^2-8}{4} + \frac{5}{14}(\epsilon-1) = \frac{\pi^2-8}{4} + \frac{5}{14}\left(\sqrt{1-\frac{m}{k}\omega^2}-1\right) \approx \frac{\pi^2-8}{4} - \frac{5m\omega^2}{28k}$$

As we expect, the variance decreases as we increase ω. This approximation breaks down for large values of $\omega^2 \ \frac{k}{m}$, which we expect the variance to be around 0.

Because it wasn't specified whether the potential was per unit mass or not, points were given to solutions that treated U as per unit mass. Since the first order wasn't explicitly specified, we gave points to attempted solutions that were close to the actual value.

Grading Scheme: **5.0 pt** total.

- 2 pt for the correct integral/setup for variance in inclination. (-0.5 for sign and algebraic errors)
- 1 pt for obtaining the variance for $\omega^2 \ll m/k$ in 0th order.
- 1 pt for using an approximation.
- 1 pt for correct expression.

4 Hot Solids

In this problem, we investigate a one-dimensional model of atoms in a solid. Assume the atoms are point masses of mass m connected by springs with spring constant κ and rest length a, and the total rest length of the chain is L.

4.1

First, assume that the mass is spread continuously throughout the chain (in other words, a is very small). Here, longitudinal waves have the same speed for all values of the angular frequency ω and wavenumber k.

(a) Find this speed of sound in the solid, v, up to a dimensionless constant.

Now, we get rid of this assumption and solve fully.

(b) Find a dispersion relation (a relationship between ω and k) for the chain of atoms if a is not required to be small. Use this result to find the dimensionless constant from part (a).

4.2

We can use the above results to find the heat capacity of the chain. To do so, treat each possible frequency ω as its own quantum harmonic oscillator (QHO) with a particle of mass m moving in a potential defined by $V(x) = \frac{1}{2}m\omega^2 x^2$. Each of these harmonic oscillators is at thermal equilibrium, and the total energy of the chain is the sum of the contributions from each frequency. You may find the following integrals useful:

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \frac{\pi^2}{6}, \quad \int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}$$

(c) First, derive the energy levels of a quantum harmonic oscillator by using the *WKB approximation*:

$$\oint p(x)\, dx = 2\pi\hbar(n + 1/2) \tag{1}$$

Here, $p(x)$ is the momentum of the particle as a function of position and the integral is across one classical period.

(d) Using the model from part (a), derive the total energy and heat capacity as a function of the temperature T. (Your result only needs to hold for $\beta\hbar\omega_{avg} \gg 1$, with $\beta = 1/k_B T$.) Assume that the atoms at either end of the chain must remain fixed in place.

(e) Using the dispersion relation from part (b), find the energy and heat capacity to the next order in T.

(f) Above we assumed $\beta\hbar\omega_{avg} \gg 1$. Why do our results fail for high T?

4.3

When the mass-energy of a particle is small compared to its energy level, relativistic corrections are required. The relativistic energy levels of a particle in a harmonic oscillator potential are given by:

$$E_n = mc^2 \left(-1 + \sqrt{1 + \frac{2\hbar\omega}{mc^2}\left(n + \frac{1}{2}\right)} \right) \tag{2}$$

(g) Use the given energy levels to find the total energy and heat capacity of the chain where each particle is moving relativistically; you may assume that the dispersion relation is linear as in part (d). Give your answer to the lowest order in $\hbar\omega/mc^2$.

Hot Solids – Solution

(a) By dimensional analysis,

$$v \sim a\sqrt{\frac{\kappa}{m}} \tag{3}$$

(b) Let the displacement of the nth mass be ψ_n. Then

$$m\ddot{\psi_n} = \kappa((\psi_{n+1} - \psi_n) - (\psi_n - \psi_{n-1})) \tag{4}$$

$$\implies m\ddot{\psi_n} = \kappa(\psi_{n+1} + psi_{n-1} - 2\psi_n) \tag{5}$$

We guess that the solution has the form $\psi(x,t) = A\exp(i(\omega t - kx))$ with x being the equilibrium positions of the masses. Then

$$-m\omega^2 A e^{i(\omega t - kx)} = \kappa A e^{i\omega t}(e^{-ik(x_n+a)} + e^{-ik(X_n-a)} - 2e^{ikx_n}) \tag{6}$$

$$\implies m\omega^2 = 2\kappa[1-\cos(ka)] \tag{7}$$

$$\implies \omega = 2\sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{ka}{2}\right)\right| \tag{8}$$

In the limit of small a, we get that the dimensionless constant from (a) is 1.
(c)

$$2\int_{-L}^{L} m\omega\sqrt{L^2 - x^2}\mathrm{d}x = 2\pi\hbar(n+1/2) \tag{9}$$

$$\implies \pi L^2 m\omega = 2\pi\hbar(n+1/2) \tag{10}$$

$$\implies E = \frac{1}{2}m\omega^2 L^2 = \hbar\omega(n+1/2) \tag{11}$$

(d) Each sound mode with wavenumber k has energy in the associated modes with frequency ω. To derive the energy in a mode with frequency ω, we first find the partition function for the harmonic oscillator:

$$Z = \sum_{n=0}^{\infty} \exp(-\beta\hbar\omega(n+1/2)) = \frac{\exp\left(-\frac{1}{2}\beta\hbar\omega\right)}{1-\exp(-\beta\hbar\omega)} \tag{12}$$

The total energy is given by $-\partial_\beta \ln Z$, which gives

$$E_\omega = \frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1} \tag{13}$$

The total energy, with $\omega(k) = ak\sqrt{\frac{\kappa}{m}}$

$$E = \sum E_\omega \tag{14}$$

$$\implies = \frac{L}{\pi}\int_0^\infty \mathrm{d}k\, \frac{\hbar\omega(k)}{\exp(\beta\hbar\omega(k)) - 1} \tag{15}$$

$$\implies = \frac{L}{\pi}\int_0^\infty \mathrm{d}\omega\, \frac{1}{a}\sqrt{\frac{m}{\kappa}}\frac{\hbar\omega}{\exp(\beta\hbar\omega) - 1} \tag{16}$$

$$\implies = \frac{L}{\pi a\beta^2\hbar}\sqrt{\frac{m}{\kappa}}\int_0^\infty \frac{x}{e^x - 1}\mathrm{d}x \tag{17}$$

$$\implies = \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}} = \frac{\pi L k_B^2 T^2}{6a\hbar}\sqrt{\frac{m}{\kappa}} \tag{18}$$

Where we replace the sum with an integral,

$$\sum \longrightarrow \frac{L}{\pi}\int \tag{19}$$

This is because the two ends are fixed, so we know that $n\lambda = 2L$ for a mode of wavelength λ. Since $\lambda = 2\pi/k$, we have that $k = n\pi/L$ and therefore the spacing between k values is π/L. An integral is approximated as this spacing times the value of the function for each k value, so we can make the replacement as above. Finally, we compute

$$C_V = \frac{\pi L k_B^2 T}{3a\hbar}\sqrt{\frac{m}{\kappa}} \tag{20}$$

(e) Let $v_0 = a\sqrt{\frac{\kappa}{m}}$. Then

$$\omega = \frac{2v_0}{a}\sin\left(\frac{ka}{2}\right) \tag{21}$$

$$\implies \arcsin\left(\frac{a\omega}{2v_0}\right) = \frac{ka}{2} \tag{22}$$

$$\implies k \approx \frac{a}{v_0}\left(1 + \frac{a^2\omega^2}{24v_0^2}\right) \tag{23}$$

$$\implies \mathrm{d}k = \frac{\mathrm{d}\omega}{v_0}\left(1 + \frac{\omega^2 a^2}{8v_0^2}\right) \tag{24}$$

As above, we sum up E_ω as an integral:

$$E = \frac{L}{\pi v_0}\int_0^\infty \mathrm{d}\omega\, \frac{\hbar\omega\left(1 + \frac{\omega^2 a^2}{8v_0^2}\right)}{e^{\beta\hbar\omega} - 1} \tag{25}$$

$$\implies E = \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}} + \frac{L}{\pi v_0}\int_0^\infty \mathrm{d}\omega\, \frac{\frac{\hbar\omega^3 a^2}{8v_0^2}}{e^{\beta\hbar\omega} - 1} \tag{26}$$

$$\implies E = \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}} + \frac{La^2}{8\pi\beta^4 v_0^3\hbar^3}\int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x \tag{27}$$

$$\implies E = \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}}\left(1 + \frac{\pi^2 m}{20\beta^2\hbar^2\kappa}\right) \tag{28}$$

$$\implies E = \frac{\pi L k_B^2 T^2}{6a\hbar}\sqrt{\frac{m}{\kappa}}\left(1 + \frac{\pi^2 m k_B^2 T^2}{20\hbar^2\kappa}\right) \tag{29}$$

And finally, we get $C_V = \partial_\beta E$:

$$C_V = \frac{\pi L k_B^2 T}{3a\hbar}\sqrt{\frac{m}{\kappa}}\left(1 + \frac{\pi^2 m k_B^2 T^2}{10\hbar^2\kappa}\right) \tag{30}$$

(f) Our integral bounds go to infinity, but there are a finite number of of states. This is important at high temperatures, but for low temperatures the occupation of higher states is so low that we can make this approximation.

(g) Credit to SSHS for their awesome solution to this part! It was better than the current official solution so we present their approach here.

We find the partition function using energy values to first order in the relativistic correction:

$$Z \approx \sum_{n=0}^{\infty}\exp\left(-\beta\hbar\omega\left(n + \frac{1}{2}\right) + \frac{\beta\hbar^2\omega^2}{2mc^2}\left(n + \frac{1}{2}\right)^2\right) \tag{31}$$

$$\approx \sum_{n=0}^{\infty}\exp\left(-\beta\hbar\omega\left(n + \frac{1}{2}\right)\right)\left(1 + \frac{\beta\hbar^2\omega^2}{2mc^2}\left(n + \frac{1}{2}\right)^2\right) \tag{32}$$

$$= \left(1 + \frac{\beta}{2mc^2}\frac{\partial^2}{\partial\beta^2}\right)\sum_{n=0}^{\infty}\exp\left(-\beta\hbar\omega\left(n + \frac{1}{2}\right)\right) \tag{33}$$

Since we know that the right sum is the partition function for the classical case as above, we can go ahead and compute

$$Z = \frac{e^{\frac{1}{2}\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} + \frac{\hbar^2\omega^2}{8mc^2}\frac{\beta e^{\frac{1}{2}\beta\hbar\omega}(6e^{\beta\hbar\omega} + e^{2\beta\hbar\omega} + 1)}{(e^{\beta\hbar\omega} - 1)^3} \tag{34}$$

So

$$E_\omega = -\frac{\partial}{\partial\beta}\ln Z \tag{35}$$
$$= -\frac{\partial}{\partial\beta}\ln\left[\left(\frac{e^{\frac{1}{2}\beta\hbar\omega}}{e^{\beta\hbar\omega}-1}\right)\left(1+\frac{\hbar^2\omega^2}{8mc^2}\frac{\beta(6e^{\beta\hbar\omega}+e^{2\beta\hbar\omega}+1)}{(e^{\beta\hbar\omega}-1)^2}\right)\right] \tag{36}$$
$$= -\frac{\partial}{\partial\beta}\left[\ln\left(\frac{e^{\frac{1}{2}\beta\hbar\omega}}{e^{\beta\hbar\omega}-1}\right)+\ln\left(1+\frac{\hbar^2\omega^2}{8mc^2}\frac{\beta(6e^{\beta\hbar\omega}+e^{2\beta\hbar\omega}+1)}{(e^{\beta\hbar\omega}-1)^2}\right)\right] \tag{37}$$
$$\approx -\frac{\partial}{\partial\beta}\left[\ln\left(\frac{e^{\frac{1}{2}\beta\hbar\omega}}{e^{\beta\hbar\omega}-1}\right)+\frac{\hbar^2\omega^2}{8mc^2}\frac{\beta(6e^{\beta\hbar\omega}+e^{2\beta\hbar\omega}+1)}{(e^{\beta\hbar\omega}-1)^2}\right] \tag{38}$$
$$= \frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}+\frac{\hbar^2\omega^2}{8mc^2}\left[\frac{16\beta\hbar\omega}{(e^{\beta\hbar\omega}-1)^3}+\frac{8(\beta\hbar\omega-1)}{e^{\beta\hbar\omega}-1}+\frac{8(3\beta\hbar\omega-1)}{(e^{\beta\hbar\omega}-1)^2}-1\right] \tag{39}$$

Now, we integrate to find the total energy (dropping the zero point energy, of course):

$$E = \sum E_\omega \tag{40}$$
$$= \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}} \tag{41}$$
$$+ \frac{\hbar^2 L}{8\pi mc^2 a}\sqrt{\frac{m}{\kappa}}\int \mathrm{d}\omega\ \omega^2\left[\frac{16\beta\hbar\omega}{(e^{\beta\hbar\omega}-1)^3}+\frac{8(\beta\hbar\omega-1)}{e^{\beta\hbar\omega}-1}+\frac{8(3\beta\hbar\omega-1)}{(e^{\beta\hbar\omega}-1)^2}-1\right] \tag{42}$$
$$= \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}} \tag{43}$$
$$+ \frac{L}{\pi mc^2\beta^3 a\hbar}\sqrt{\frac{m}{\kappa}}\left[2\int_0^\infty \mathrm{d}x\ \frac{x^3}{e^x-1}+\int_0^\infty \mathrm{d}x\ \frac{x^3}{e^x-1}-\int_0^\infty \mathrm{d}x\ \frac{x^2}{e^x-1}\right. \tag{44}$$
$$\left.+3\int_0^\infty \mathrm{d}x\ \frac{x^3}{(e^x-1)^2}-\int_0^\infty \mathrm{d}x\ \frac{x^2}{(e^x-1)^2}\right] \tag{45}$$
$$= \frac{\pi L}{6a\beta^2\hbar}\sqrt{\frac{m}{\kappa}}+\frac{L}{\pi mc^2\beta^3 a\hbar}\sqrt{\frac{m}{\kappa}}\ [\pi^2-2\zeta(3)-0.88574] \tag{46}$$

Which, finally, gives us

$$C_V = \frac{L\pi k_B^2 T}{3\hbar a}\sqrt{\frac{m}{\kappa}}\left[1+\frac{9k_BT}{\pi^2 mc^2}(\pi^2-2\zeta(3)-0.88574)\right] \tag{47}$$

There are more ways to do this that give slightly different results due to the approximations made. Some ways are replacing the sum in the partition function with an integral or computing the total energy directly as an integral (without finding the partition function first and taking a derivative).