2023 Online Physics Olympiad: Open Contest

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.8$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in base SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before July 25, 2023**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J} \cdot \mathrm{s} = 4.41 \cdot 10^{-15}\ \mathrm{eV} \cdot \mathrm{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C}^2/(\mathrm{N} \cdot \mathrm{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T} \cdot \mathrm{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T} \cdot \mathrm{m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^{5}\ \mathrm{N/m^2} = 1.01 \cdot 10^{5}\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Problems

We thank contestants (in no order) Baiyu Zhu, David Lee, Ong Zhi Zheng, Guangyuan Chen, and Nathan Zhao for contributing some solutions.

1. Coin Flip 1 The coin flip has long been recognized as a simple and unbiased method to randomly determine the outcome of an event. In the case of an ideal coin, it is well-established that each flip has an equal 50% chance of landing as either heads or tails.

However, coin flips are not entirely random. They appear random to us because we lack sufficient information about the coin's initial conditions. If we possessed this information, we would always be able to predict the outcome without needing to flip the coin. For an intriguing discussion on why this observation is significant, watch this video by Vsauce.

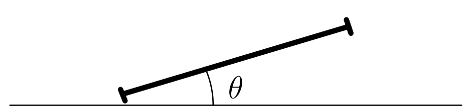
Now, consider a scenario where a coin with uniform density and negligible width is tossed directly upward from a height of $h = 0.75$ m above the ground. The coin starts with its heads facing upward and is given an initial vertical velocity of $v_y = 49$ m/s and a positive angular velocity of $\omega = \pi$ rad/s. What face does the coin display upon hitting the ground? **Submit 0 for heads and 1 for tails.** You only have one attempt for this problem. Assume the floor is padded and it absorbs all of the coin's energy upon contact. The radius of the coin is negligible.

Solution 1: We have the following quadratic:

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ 0 &= 0.75 + 49t - 4.9t^2 \\ t &= -0.01, 10.01 \end{aligned}$$

The first solution is extraneous so $t = 10.01$ is correct. Now, $\theta = \omega t \approx 10\pi$. As one full rotation is $\phi = 2\pi$, then the coin performs 5 full rotations before landing on the ground. This means the answer is $\boxed{0}$, or heads.

2. Coin Flip 2 A coin of uniform mass density with a radius of $r = 1$ cm is initially at rest and is released from a slight tilt of $\theta = 8°$ onto a horizontal surface with an infinite coefficient of static friction. The coin has a thicker rim, allowing it to drop and rotate on one point. With every collision, the coin switches pivot points on the rim, and energy is dissipated through heat so that $k = 0.9$ of the coin's prior total energy is conserved. How long will it take for the coin to come to a complete stop?



A cross-sectional view of the coin before release. The rim can be seen on the edges of the coin.

Solution 2: By the parallel axis theorem, the moment of inertia of the coin around the pivot can be expressed as

$$I = I_x + m\ell^2 = \frac{1}{4}mr^2 + mr^2 = \frac{5}{4}mr^2.$$

As θ is small, that means the perpendicular force of gravity is $mg\cos\theta \approx mg$. Hence, Newton's second law to find the angular acceleration as

$$I\alpha = \tau \implies \alpha = \frac{4g}{5r} \cdot (1 \text{ rad})$$

Using rotational kinematics, the time for the first collision will be $\theta = \frac{1}{2}\alpha t^2 \implies t_0 = \sqrt{\frac{5r\theta_0}{2g}}$. By conservation of energy, the angular velocity at the time of collision is $\frac{1}{2}I\omega_0^2 = mgr\sin\theta \approx mgr$, meaning that $\omega_0 = \sqrt{\frac{8g\theta_0}{5r}}$. Now consider k. This is the ratio of the initial energy E_{n-1} and next energy E_n. Then, we can say that

$$k = \frac{\frac{1}{2}I\omega_n^2}{\frac{1}{2}I\omega_{n-1}^2} \implies \omega_n = \sqrt{k}\omega_{n-1}.$$

By recurrence, $\omega_n = k^{n/2}\omega_0$. The time of flight for each cycle will be $t_n = \frac{2\omega_n}{\alpha}$. Therefore,

$$\begin{aligned} T &= t_0 + \sum_{n=1}^{\infty} t_n \\ &= \sqrt{\frac{5r\theta_0}{2g}} + \sum_{n=1}^{\infty}\left(2k^{n/2}\sqrt{\frac{8g\theta_0}{5r}} \cdot \frac{5r}{4g}\right) \\ &= \sqrt{\frac{5r\theta_0}{2g}} + \sqrt{\frac{10r\theta_0}{g}}\frac{k^{1/2}}{1-k^{1/2}} \end{aligned}$$

Converting 8° to $\frac{2\pi}{45}$ radians, $k = 0.9$, $g = 9.8$ m/s^2, $r = 0.01$ m, we find that $\boxed{T = 0.716 \text{ s}}$ which is a reasonable estimate.

Alternate: This problem can also be solved by creating a differential equation for the height h of the center of mass of the coin and finding the time for $h \to 0$.

3. Highway Suppose all cars on a (single-lane) highway are identical. Their length is $l = 4$m, their wheels have coefficients of friction $\mu = 0.7$, and they all travel at speed v_0. Find the v_0 which maximizes the flow rate of cars (i.e. how many cars travel across an imaginary line per minute). Assume that they need to be able to stop in time if the car in front instantaneously stops. Disregard reaction time.

Solution 3: Suppose the maximum speed is v'. Notice that $a = \mu g$ so it takes a time $t' = \frac{v'}{\mu g}$ amount of time to stop. This means the car will travel a distance of

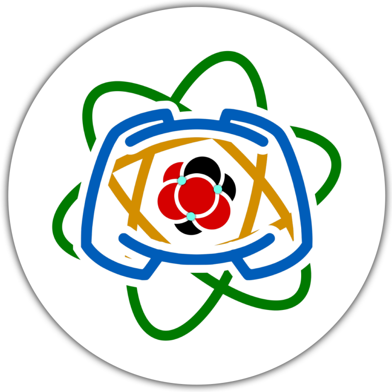
$$d = \frac{1}{2}\mu g\left(\frac{v'}{\mu g}\right)^2 = \frac{v'^2}{2\mu g}.$$

Thus for every distance $d + l = l + \frac{v'^2}{2\mu g}$ there is a car. This means in unit time t, there will be $N = \frac{d+l}{v}$ cars that passes through the line. To maximize the flow rate, we need to maximize

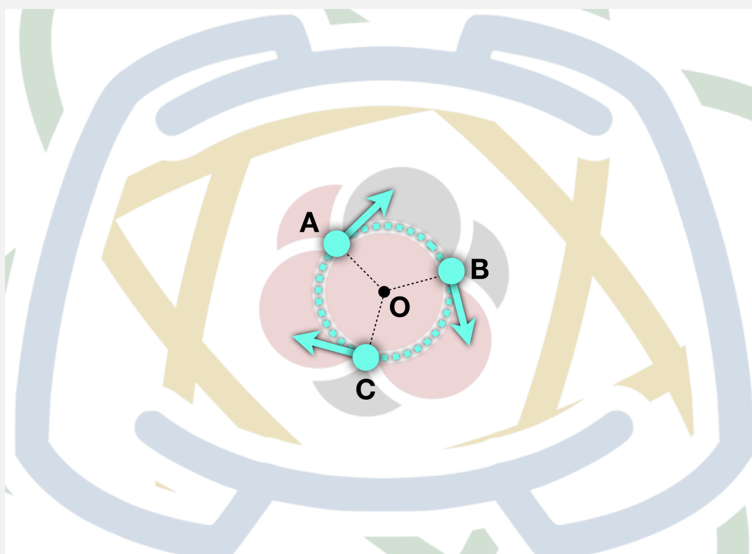
$$f(v') = \frac{l}{v'} + \frac{v'}{2\mu g}$$

$$f' = \frac{-l}{v'^2} + \frac{1}{2\mu g} = 0 \Rightarrow v' = \sqrt{2\mu g l} = \boxed{7.41 \text{ m/s}}$$

4. Spinning Around Here is a Physoly round button badge, in which the logo is printed on the flat and rigid surface of this badge. Toss it in the air and track the motions of three points (indicated by cyan circles in the figure) separated a straight-line distance of $L = 5$ mm apart. At a particular moment, we find that these all have the same speed $V = 4$ cm/s but are heading to different directions which form an angle of $\theta = 30°$ between each pair. Determine the then angular velocity of the badge (in rad/s).



Solution 4: Call the three tracking points on the Physoly badge A, B, C, and their geometrical center O. The distance from O to these three points are the same and equal to $L/\sqrt{3}$.



Due to symmetry, the velocity vector of O has to be perpendicular to the ABC plane. In the reference frame of O, the points A, B, C both have the same speed $2V\sin(\theta/2)/\sqrt{3}$ but are heading to different directions which form an angle of 120° between each pair. Also due to symmetry, the axis of rotation has to be perpendicular to the ABC plane, thus the velocity vectors of points A, B, C in O reference frame looks like described in the attached figure. For $L = 5$mm, $V = 4$cm/s, $\theta = 30° = \pi/12$, the angular velocity of the badge can be calculated as:

$$\Omega = \frac{2V\sin(\theta/2)/\sqrt{3}}{L/\sqrt{3}} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)\frac{V}{L} \approx \boxed{4.1411\text{rad/s}}$$

5. Born To Try In a resource-limited ecological system, a population of organisms cannot keep growing forever (such as lab bacteria growing inside culture tube). The effective growth rate g (including contributions from births and deaths) depends on the instantaneous abundance of resource $R(t)$, which in this problem we will consider the simple case of linear-dependency:

$$\frac{\mathrm{d}}{\mathrm{d}t}N = g(R)N = \alpha RN \ ,$$

where $N(t)$ is the population size at time t. The resources is consumed at a constant rate β by each organism:

$$\frac{\mathrm{d}}{\mathrm{d}t}R = -\beta N \ .$$

Initially, the total amount of resources is R_0 and the population size is N_0. Given that $\alpha = 10^{-9}$ resource-unit^{-1}s^{-1}, $\beta = 1$ resource-unit/s, $R_0 = 10^6$ resource-units and $N_0 = 1$ cell, find the total time it takes from the beginning to when all resources are depleted (in hours).

Solution 5:
We can find the analytical solution for the following set of two ODEs describing the population-resource dynamics:

$$\frac{dN}{dt} = \alpha RN \ , \tag{1}$$

$$\frac{dR}{dt} = -\beta N \ . \tag{2}$$

Divide Eq.(1) by Eq.(2) on both sides, we get a direct relation between the population size $N(t)$ and the amount of resource $R(t)$:

$$\frac{dN}{dR} = -\frac{\alpha}{\beta}R \implies N = N_0 + \frac{\alpha}{2\beta}\left(R_0^2 - R^2\right) \ .$$

Plug this in Eq.(2), we obtain the total time T it takes from the beginning to when the resource is depleted:

$$\frac{dR}{dt} = -\frac{\alpha}{2}\left[\left(\frac{2\beta}{\alpha}N_0 + R_0^2\right) - R^2\right] \implies t\Big|_{R=0} = \frac{2}{\alpha}\int_0^{R_0} dR\left[\left(\frac{2\beta}{\alpha}N_0 + R_0^2\right) - R^2\right]^{-1}$$

$$= \frac{2}{\alpha\sqrt{\frac{2\beta}{\alpha}N_0 + R_0^2}}\operatorname{arctanh}\left(\frac{R_0}{\sqrt{\frac{2\beta}{\alpha}N_0 + R_0^2}}\right) \ .$$

Use the given numerical values, we arrive at $t\Big|_{R=0} \approx 7594.3\text{s} \approx \boxed{2.1095\text{hrs}}$.

6. Lightbulb An incandescent lightbulb is connected to a circuit which delivers a maximum power of 10 Watts. The filament of the lightbulb is made of Tungsten and conducts electricity to produce light. The specific heat of Tungsten is $c = 235$ J/(K $\cdot$ kg). If the circuit is alternating such that the temperature inside the lightbulb fluctuates between $T_0 = 3000°$ C and $T_1 = 3200°$ C at a frequency of $\omega = 0.02$ s^{-1}, estimate the mass of the filament.

Solution 6: This problem was voided from the test because we gave no method for energy dissipation. Therefore, there was ambiguity and energy would be constantly fed to the lightbulb making it hotter and hotter.

However, the problem can still be solved. Here is a solution written by one of our contestants

Guangyuan Chen. The circuit continuously delivers sinusoidal power to the lightbulb. In order to maintain a quasi-equilibrium state, there must be some form of heat loss present in the system. We will assume that the main source of heat loss is through radiation, though we will see that the precise form of heat loss is largely inconsequential.

Let the power delivered by the circuit be in the form $P_d = P_0 \sin^2 \Omega t$. Let the power loss due to the radiation be $P_l = -kT^4$ where k is some constant and T is the temperature of the filament. We can write:

$$mc\frac{\mathrm{d}T}{\mathrm{d}t} = P_0 \sin^2 \Omega t - kT^4$$

This is a differential equation that cannot be solved by hand. However, since the deviation in temperature from the equilibrium of roughly 100 °C is much smaller than the equilibrium temperature of roughly 3100 °C, we can do a first order approximation of the differential equation. The temperature thus varies sinusoidally with equilibrium temperature $T_{equi} = 3100\,^\circ\mathrm{C}$ and amplitude $T_a = 100\,^\circ\mathrm{C}$. Since the average power delivered is $\frac{1}{2}P_0$, at equilibrium temperature, we have:

$$\frac{1}{2}P_0 = kT_{equi}^4$$

$$k = \frac{P_0}{2T_{equi}^4}$$

We rewrite equation 1 with the approximation $T = T_{equi} + \Delta T$ and with the substitution from equation 3.

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} = P_0 \sin^2 \Omega t - \frac{P_0}{2T_{equi}^4}(T_{equi} + \Delta T)^4$$

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} \approx P_0 \sin^2 \Omega t - \frac{P_0}{2}\left(1 + 4\frac{\Delta T}{T_{equi}}\right)$$

$$mc\frac{\mathrm{d}\Delta T}{\mathrm{d}t} \approx -\frac{P_0}{2}\left(\cos 2\Omega t + 4\frac{\Delta T}{T_{equi}}\right)$$

At this point we can see that regardless of the physical mechanism of heat loss, a linear approximation can always be made for sufficiently small variations in temperature which results in an equation of the same form as equation 6. To solve this equation, we can guess the following solution:

$$\Delta T = A \sin 2\Omega t + B \cos 2\Omega t$$

Strictly speaking, we also need to include a term $Ce^{-\lambda t}$. However, for sufficiently long times approaching quasi-equilibrium, this term will go to zero, so we need not include it here. Substituting equation 7 into equation 6:

$$2mc\Omega(A\cos 2\Omega t - B \sin 2\Omega t) = -\frac{P_0}{2}\left(\cos 2\Omega t + \frac{4}{T_{equi}}(A \sin 2\Omega t + B \cos 2\Omega t)\right)$$

Equating the coefficients of the sin and cos:

$$2mc\Omega A = -\frac{P_0}{2}\left(1 + \frac{4B}{T_{equi}}\right)$$

$$-2mc\Omega B = -\frac{2P_0 A}{T_{equi}}$$

Equations 9 and 10 can be solved simultaneously to give

$$A = -\frac{mc\Omega T_{equi} P_0}{4((mc\Omega T_{equi})^2 + P_0^2)} T_{equi}$$

$$B = -\frac{P_0^2}{4((mc\Omega T_{equi})^2 + P_0^2)} T_{equi}$$

Equation 7 can be written in the form $T_a \sin(2\Omega t + \phi)$. To find T_a, we simply have:

$$T_a = \sqrt{A^2 + B^2} = \frac{P_0}{4\sqrt{(mc\Omega T_{equi})^2 + P_0^2}} T_{equi}$$

Rearranging for m, we have:

$$m = \frac{P_0}{c\Omega T_{equi}} \sqrt{\left(\frac{T_{equi}}{4T_a}\right)^2 - 1}$$

The final step is to relate Ω and ω. Since T oscillates with angular frequency 2Ω, its period of oscillation is $\frac{2\pi}{2\Omega}$. This is equal to $\frac{1}{\omega}$, hence we have:

$$\Omega = \pi\omega$$

Substituting into equation 14, we get our final numerical answer.

$$m = \frac{P_0}{\pi c\omega T_{equi}} \sqrt{\left(\frac{T_{equi}}{4T_a}\right)^2 - 1} = \boxed{1.68 \times 10^{-3}\,\text{kg}}$$

7. Hyperdrive In hyperdrive, Spaceship-0 is relativistically moving at the velocity $\frac{1}{3}c$ with respect to reference frame R_1, as measured by Spaceship-1. Spaceship-1 is moving at $\frac{1}{2}c$ with respect to reference frame R_2, as measured by Spaceship-2. Spaceship-k is moving at speed $v_k = \frac{k+1}{k+3}c$ with respect to reference frame R_{k+1}. The speed of Spaceship-0 with respect to reference frame R_{20} can be expressed as a decimal fraction of the speed of light which has only x number of 9s following the decimal point (i.e., in the form of $0.\underbrace{99\ldots9}_{x \text{ times}}c$). Find the value of x.

Solution 7: Let us define the rapidity as

$$\tanh\phi \equiv \beta = \frac{v}{c}$$

where tanh is the hyperbolic tangent function. Let $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$. If we add β_1 and β_2 using the relativistic velocity addition formula, we find that

$$\beta = \frac{\beta_1 + \beta_2}{1 - \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} = \tanh(\phi_1 + \phi_2).$$

We can then rewrite the problem as

$$v_f = \tanh\left(\operatorname{arctanh}\frac{1}{3} + \operatorname{arctanh}\frac{2}{4} + \cdots + \operatorname{arctanh}\frac{20}{22}\right).$$

Using the fact that $\text{arctanh}(\phi) = \frac{1}{2}\ln\left(\frac{1+\phi}{1-\phi}\right)$, we can find that

$$v_f = \tanh\left(\frac{1}{2}\sum_{k=0}^{19}\ln\left(\frac{1+\frac{k+1}{k+3}}{1-\frac{k+1}{k+3}}\right)\right) = \tanh\left(\frac{1}{2}\sum_{k=0}^{19}\ln(k+2)\right)$$
$$= \tanh\left(\ln\sqrt{2\cdot 3\cdot 4\cdots 21}\right) = \tanh\left(\ln\sqrt{21!}\right)$$

As $\tanh\phi = (e^{\phi} - e^{-\phi})/(e^{\phi}+e^{-\phi})$, then

$$\tanh(\ln(\phi)) = \frac{\phi - \frac{1}{\phi}}{\phi + \frac{1}{\phi}} = 1 - \frac{2}{\phi^2+1} \implies v_f = 1 - \frac{2}{21!+1}$$

This implies $\boxed{19}$ zeros, but you can also use Stirlings approximation to further approximate the factorial.

Alternate 1: Define u_k as Spaceship 0's velocity in frame R_k, given $c=1$. The recurrence relation from the relativistic velocity addition formula is: $u_{k+1} = \frac{u_k(k+3)+(k+1)}{u_k(k+1)+(k+3)}$, starting with $u_1 = \frac{1}{3}$. The relation can be simplified as: $u_{k+1} = 1 + \frac{2(u_k-1)}{u_k(k+1)+(k+3)}$. Introducing $v_k = u_k - 1$, $v_{k+1} = v_k\frac{2}{v_k(k+1)+(2k+4)}$. Further simplifying with $w_k = \frac{1}{v_k}$, we get $w_{k+1} = w_k(k+2) + \frac{k+1}{2}$. By setting $x_k = w_k + c$, we find $c = -\frac{1}{2}$ simplifies to $x_{k+1} = x_k(k+2)$. This gives $x_k = (k+1)(k)(k-1)...(4)(3)x_1$ and using the initial condition $x_1 = \frac{1}{u_1-1} + \frac{1}{2} = 1$, we obtain $x_k = \frac{(k+1)!}{2}$. Consequently, $u_k = \frac{(k+1)!-1}{(k+1)!+1}$ and substituting $k=20$, $u_{20} \approx 1 - 3.9\times 10^{-20}$, yielding $\boxed{19}$ significant digits.

Alternate 2: Let $l = k+2$, then $u_l = \frac{l-1}{l+1}$. Let $u_m = \frac{m-1}{m+1}$. Then you can find that the velocity addition of any l, m will be $\frac{ml-1}{ml+1}$. Using this identity, we can use recurrence to find that $u_k = \frac{(k+1)!-1}{(k+1)!+1}$.

8. Asteroid The path of an asteroid that comes close to the Earth can be modeled as follows: neglect gravitational effects due to other bodies, and assume the asteroid comes in from far away with some speed v and lever arm distance r to Earth's center. On January 26, 2023, a small asteroid called 2023 BU came to a close distance of 3541 km to Earth's surface with a speed of 9300 m/s. Although BU had a very small mass estimated to be about 300,000 kg, if it was much more massive, it could have hit the Earth. How massive would BU have had to have been to make contact with the Earth? Express your answer in scientific notation with 3 significant digits. Use 6357 km as the radius of the Earth. The parameters that remain constant when the asteroid mass changes are v and r, where v is the speed at infinity and r is the impact parameter.

Solution 8:
Let $v_1 = 9300$ m/s, $d = 3541$ km, and $m = 300,000$ kg, and let M and R be the Earth's mass and radius.
First we find v and r. We use the reference frame of the Earth, where the asteroid has reduced mass $\mu = \frac{Mm}{M+m}$ and the Earth has mass $M+m$. Then by energy and angular momentum conservation, we have

$$\mu v r = \mu v_1 (R+d)$$

and

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu v_1^2 - \frac{GMm}{R+d}.$$

We solve for

$$v = \sqrt{2G(M+m) \cdot \frac{R+d}{r^2 - (R+d)^2}},$$

so

$$v_1 = \sqrt{\frac{2G(M+m)}{R+d} \cdot \frac{r^2}{r^2 - (R+d)^2}},$$

and we compute $r = 37047$ km and $v = 2490$ m/s.
Now we consider when the asteroid is massive enough to touch the Earth. We let m' and μ' be the mass of the asteroid and its reduced mass, and using a similar method to above, we arrive at

$$v = \sqrt{2G(M+m') \cdot \frac{R}{r^2 - R^2}},$$

so we can solve for $m' = 3.74 \times 10^{24}$ kg.

9. Spaceship IK Pegasi and Betelgeuse are two star systems that can undergo a supernova. Betelgeuse is 548 light-years away from Earth and IK Pegasi is 154 light-years away from Earth. Assume that the two star systems are 500 light-years away from each other.

Astronomers on Earth observe that the two star systems undergo a supernova explosion 300 years apart. A spaceship, the *OPhO Galaxia Explorer* which left Earth in an unknown direction before the first supernova observes both explosions occur simultaneously. Assume that this spaceship travels in a straight line at a constant speed v. How far are the two star systems according to the *OPhO Galaxia Explorer* at the moment of the simultaneous supernovae? Answer in light-years.

Note: Like standard relativity problems, we are assuming intelligent observers that know the finite speed of light and correct for it.

Solution 9: For any inertial observer, define the 4-distance between two events as $\Delta s^\mu = (c\Delta t, \Delta \mathbf{x})$, where Δt and $\Delta \mathbf{x}$ are the temporal and spatial intervals measured by the observer. By the properties of 4-vectors, the following quantity is Lorentz-invariant:

$$\Delta s^\mu \Delta s_\mu = c^2 \Delta t^2 - \|\Delta \mathbf{x}\|^2$$

In the spaceship's frame, this is equal to $\Delta s^\mu \Delta s_\mu = -\|\Delta \mathbf{x}'\|^2$, since the two supernovas are simultaneous; hence

$$\|\Delta \mathbf{x}'\| = \sqrt{\|\Delta \mathbf{x}\|^2 - c^2 \Delta t^2}$$

Since the observers have already taken into account the delay due to the nonzero speed of light, $c\Delta t = 300$ly, $\|\Delta \mathbf{x}\| = 500$ly, and

$$\|\Delta \mathbf{x}'\| = \sqrt{\|\Delta \mathbf{x}\|^2 - c^2 \Delta t^2} = 400\text{ly}$$

10. Drag 1 A ball of mass 1 kg is thrown vertically upwards and it faces a quadratic drag with a terminal velocity of 20 m/s. It reaches a maximum height of 30 m and falls back to the ground. Calculate the energy dissipated until the point of impact (in J).

Solution 10: If we suppose that the magnitude of quadratic drag is $F_{\text{drag}} = cv^2$ for some constant c, then when the ball is falling downwards at terminal velocity v_t this upwards drag force must cancel gravity to provide zero net force:

$$mg = cv_t^2 \implies c = \frac{mg}{v_t^2}$$

so we can rewrite our equations of motion in terms of terminal velocity. In the ascending portion of the ball's trajectory, suppose that the ball's speed changes from v to $v + dv$ after traveling an infinitesimal distance from h to $h + dh$. Then:

$$d(\text{kinetic energy}) = -d(\text{potential energy and energy lost to drag}) \implies mv\,dv = -\frac{mgv^2}{v_t^2}\,dh - mg\,dh$$

where we used chain rule and $dW = F \cdot dx$. This is a separable differential equation for velocity in terms of height (!), which we can solve:

$$\int \frac{v\,dv}{1 + \frac{v^2}{v_t^2}} = \int -g\,dh + \text{constant} = \text{constant} - gh$$

We u-sub the entire denominator ($u = 1 + v^2/v_t^2 \implies du = 2v/v_t^2\,dv$), which nicely cancels the numerator. Our left-hand integral is

$$\frac{v_t^2}{2}\int \frac{du}{u} = \frac{v_t^2}{2}\ln\left|1 + \frac{v^2}{v_t^2}\right| = \text{constant} - gh$$

Now to find the constant! At our maximum height $h_0 = 30$ m, the speed and left-hand side are zero, so the constant must be gh_0. Then we can find the initial speed of the projectile by substituting $h = 0$, from which:

$$v_{\text{initial}} = v_t\sqrt{e^{\frac{2gh_0}{v_t^2}} - 1}$$

We can use a similar argument for the downwards trajectory, albeit with drag pointing upwards. We should get that the final speed immediately before impact is

$$v_{\text{final}} = v_t\sqrt{1 - e^{\frac{-2gh_0}{v_t^2}}}$$

from which the dissipated energy is the difference in final and initial kinetic energies, or

$$\frac{1}{2}m\left(v_{\text{final}}^2 - v_{\text{initial}}^2\right) \approx 515.83 \text{ joules}$$

11. Drag 2 In general, we can describe the quadratic drag on an object by the following force law:

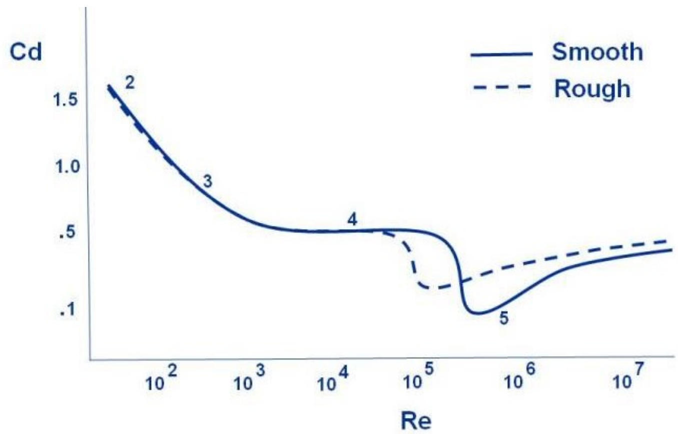
$$F_D = \frac{1}{2}C_D\rho A v^2$$

where A is the cross-sectional area of the object exposed to the airflow, v is the speed of the object in a fluid, and C_D is the drag coefficient, a dimensionless quantity that varies based on shape.

Another useful quantity to know is the Reynold's number, a dimensionless quantity that helps predict fluid flow patterns. It is given by the formula:

$$\text{Re} = \frac{\rho v L}{\mu}$$

where ρ is the density of the surrounding fluid, μ is the dynamic viscosity of the fluid, and L is a reference length parameter that varies based on each object. For a smooth [1] sphere traveling in a fluid, its diameter serves as the reference length parameter.



A logarithmic graph of C_D vs Re of a sphere from the NASA Glenn Research Center.

The relationship between the drag coefficient and the Reynold's number holds significant importance. Due to the complexity of fluid dynamics, empirical data is commonly used, as depicted in the figure provided above. Notably, the figure indicates a significant decrease in the drag coefficient around $\text{Re} \approx 4 \times 10^5$. This phenomenon, known as the drag crisis, occurs when a sphere transitions from laminar to turbulent flow, resulting in a broad wake and high drag. The table in the link below presents a range of C_d versus Re values of a smooth sphere.

Desmos table: `https://www.desmos.com/calculator/wnpkg5wnt0`

Let's consider a smooth ball with a radius of 0.2 m and a mass of 0.1 kg dropped in air with a constant density of $\rho = 1.255$ kg/m^3. It is found that at velocity 5 m/s, the Reynold's number of the ball is $3.41 \cdot 10^5$. If the ball is dropped from rest, it approaches a *stable* terminal velocity v_1. If the ball is thrown downwards with enough velocity, it will experience turbulence, and approach a *stable* terminal velocity v_2. Find $\Delta v = v_2 - v_1$. Ignore any terminal velocities found for Reynold numbers less than an order of magnitude 10^{-1}.

Note: This problem is highly idealized as it assumes the atmosphere has air of constant density and temperature. In reality, this is not true!

Solution 11: Terminal velocity exists when the net force is 0. Using $v = \frac{\mu \cdot \text{Re}}{2\rho r}$ where $L = 2r$, we find that

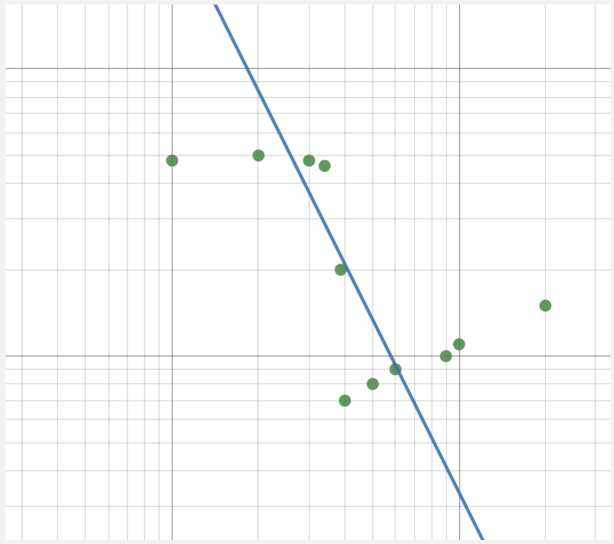
$$\frac{1}{2}\rho_a C_D(\pi r^2)\left(\frac{\mu \cdot \text{Re}}{2\rho r}\right)^2 = mg - \rho_a g\left(\frac{4}{3}\pi r^3\right).$$

[1] meaning a smooth surface.

Since $\rho = \frac{m}{4\pi r^3/3} = 2.98\ \mathrm{kg/m^3}$ is on the same order as $\rho_a = 1.255\ \mathrm{kg/m^3}$, the buoyant force must be accounted for and is non-negligible. We can rearrange to find that

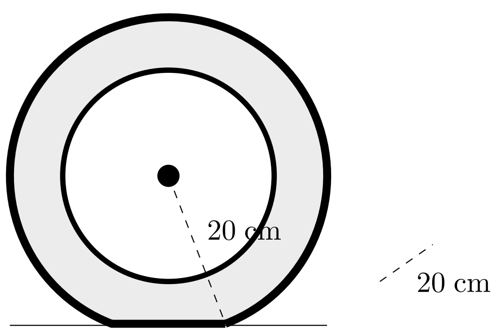
$$C_D\ \mathrm{Re}^2 = \frac{8\rho_a}{\pi\mu^2}\left(mg - \frac{4}{3}\rho_a g\pi r^3\right)$$

Using x as C_D and y as Re, we can plot an equation $xy^2 = \text{const}$ on the C_D vs Re graph. There, we can find three intersections.



The intersection in the middle is not stable. So we find the intersections of the other two to be $\mathrm{Re}_1 \approx 2.6 \times 10^5$ and $\mathrm{Re}_2 = 6 \times 10^5$. Hence, $v_1 = 3.81$ m/s and $v_2 = 8.79$ m/s, meaning $\Delta v = 4.98$ m/s. A complete working on desmos can be found here.

The following information applies for the next two problems. Pictured is a wheel from a 4-wheeled car of weight 1200kg. The absolute pressure inside the tire is 3.0×10^5Pa. Atmospheric pressure is 1.0×10^5Pa. Assume the rubber has negligible "stiffness" (i.e. a negligibly low Sheer modulus compared to its Young's modulus).



12. Hysteresis 1 The rubber on the bottom of the wheel is completely unstretched. The rubber has a thickness of 7 mm. Based on this information, find the Young's Modulus of the rubber. Is this answer reasonable?

Solution 12: Let $P = 2.0 \times 10^5$ Pa be the gauge pressure of the tire, $w = 20$ cm be the width of the tire, $r = 20$ cm be the radius of the tire, $t = 7$ mm be the thickness of the rubber, $M = 1200$ kg be the mass of the car, Y be the Young's modulus, and θ be the angle subtended by the portion of the tire in contact with the ground. We have

$$\frac{Mg}{4} = 2rwP\sin\left(\frac{\theta}{2}\right)$$

$$\theta = 2\arcsin\left(\frac{Mg}{8RwP}\right) = 0.3696$$

Let T be the tension in the stretched portion of the rubber. Balance forces on a small section of the tire subtending an angle $d\theta$.

$$Prwd\theta = 2T\sin\left(\frac{d\theta}{2}\right)$$

Using the approximation $\sin\theta = \theta$,

$$T = Prw$$

The stress on the rubber is

$$\sigma = \frac{T}{wt} = \frac{Pr}{t}$$

The strain is

$$\epsilon = \frac{\theta}{2\sin\left(\frac{\theta}{2}\right)} - 1$$

The Young's modulus is

$$Y = \frac{\sigma}{\epsilon} = \frac{\frac{Pr}{t}}{\frac{b}{2\sin\left(\frac{b}{2}\right)} - 1} = \boxed{1.00 \times 10^9 \text{ Pa}}$$

13. Hysteresis 2 The rubber experiences a phenomena known as *hysteresis* – it takes more force to stretch the rubber than allow it to return to equilibrium. Specifically, assume that the Young's Modulus when the rubber is stretched is the answer to 12, and is 1/2 of that when the rubber returns to equilibrium. Compute the power the car's engines has to deliver to overcome the hysteresis losses, if the car moves at 20 m/s. Remember that there are 4 tires!

Solution 13: From the previous part, we have the tension $T = Prw$ and strain $\epsilon = \frac{\theta}{2\sin\left(\frac{\theta}{2}\right)} - 1$ where $\theta = 2\arcsin\left(\frac{Mg}{8RwP}\right)$ is the angle subtended by the portion of the tire in contact with the road. The work required to stretch a portion of tire of length dx is

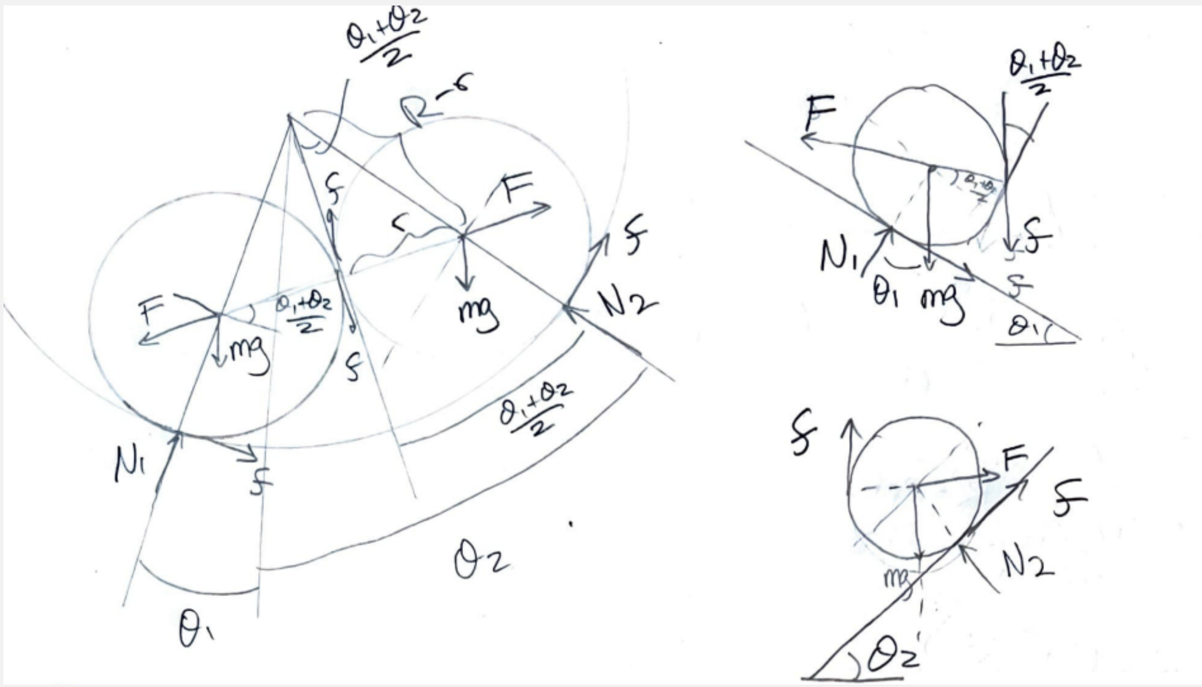
$$W = \frac{1}{2}T\epsilon dx$$

Dividing both sides by dt, multiplying by a factor of $\frac{1}{2}$ to account for half the power being restored when the rubber returns to equilibrium, and multiplying by 4 to account for all four tires, we have

$$\frac{dW}{dt} = T\epsilon\frac{dx}{dt} = T\epsilon v = \boxed{914 \text{ W}}$$

14. Marbles Two identical spherical marbles of radius 3 cm are placed in a spherical bowl of radius 10 cm. The coefficient of static friction between the two surfaces of the marble is 0.31 and the coefficient of static friction between the surfaces of the marbles and the bowl is 0.13. Find the maximum elevation from the bottom of the bowl that the center of one of the marbles can achieve in equilibrium. The bowl is fixed in place and will neither rotate nor translate. Equilibrium refers to stable equilibrium.

Solution 14:



In the figure above, $r = 3cm$ and $R = 10cm$.
By torque balance about the center of both marbles, all frictions as shown in the FBD above must be equal. WLOG, let $\theta_2 > \theta_1$ and consider the elevation of the 2nd marble for the final answer. By geometry, we can see that $\sin\frac{\theta_1+\theta_2}{2} = \frac{r}{R-r}$. Solving for equilibrium on both marbles give the following set of equations:

$$mg\sin\theta_1 + f\sin\frac{\theta_1+\theta_2}{2} + f - F\cos\frac{\theta_1+\theta_2}{2} = 0$$

$$N_1 - mg\cos\theta_1 - F\sin\frac{\theta_1+\theta_2}{2} - f\cos\frac{\theta_1+\theta_2}{2} = 0$$

$$-mg\sin\theta_2 + F\cos\frac{\theta_1+\theta_2}{2} + f + f\sin\frac{\theta_1+\theta_2}{2} = 0$$

$$N_2 - mg\cos\theta_2 - F\sin\frac{\theta_1+\theta_2}{2} + f\cos\frac{\theta_1+\theta_2}{2} = 0$$

Note that $N_1 - mg\cos\theta_1 - f\cos\frac{\theta_1+\theta_2}{2} = N_2 - mg\cos\theta_2 + f\cos\frac{\theta_1+\theta_2}{2}$ and therefore $N_2 < N_1$.
These equations can be simplified to get:

$$f = \frac{1}{2}mg\frac{\sin\theta_1 - \sin\theta_2}{1+\sin\frac{\theta_1+\theta_2}{2}}$$

$$F = \frac{1}{2}mg\frac{\sin\theta_1 + \sin\theta_2}{\cos\frac{\theta_1+\theta_2}{2}}$$

$$N_2 = \frac{1}{2}mg(2\cos\theta_2 + (\sin\theta_1 + \sin\theta_2)\cdot\tan\frac{\theta_1+\theta_2}{2} - \frac{\sin\theta_1 - \sin\theta_2}{1+\sin\frac{\theta_1+\theta_2}{2}}\cos\frac{\theta_1+\theta_2}{2})$$

There are two cases to consider. The first is that the friction between the two surfaces of the marbles is limiting, in which $f = \mu_m F$ where $\mu_m = 0.31$. The second is that the friction between the marbles and the bowl is limiting, in which $f = \mu_b F$ where $\mu_b = 0.13$. θ_2 in each case can be solved numerically, as seen here on desmos. It turns out that μ_b is the limiting factor, in which $\theta_2 = 0.6294rad$ and the elevation reached by the center of the second marble is $R-(R-r)\cos\theta_2 = 0.0434$ m.

The problem was intended to be solved with the value of $\mu_b = 0.17$. However, due to a typographical error in the problem statement, 0.13 was displayed instead. In the case of which $\mu_b = 0.17$, θ_2 has an analytical solution while it does not for 0.13. The error tolerance on this problem was adjusted such that those who submitted answers whose results derive from either a value of 0.13 and 0.17 for μ_b were accepted, in order to accommodate those less fluent with numerical software such as desmos.

When $\mu_b = 0.17$ is used instead:

$$\theta_2 = \arcsin\frac{r}{R-r} + \arctan\frac{\mu_m r}{R-2r}$$

and

$$R-(R-r)\cos\theta_2 = 0.0452 \text{ m}$$

.

15. Fringe effect approximation Two parallel square plates of side length 1 m are placed a distance 30 cm apart whose centers are at $(-15\text{ cm}, 0, 0)$ and $(15\text{ cm}, 0, 0)$ have uniform charge densities -10^{-6} C/m^2 and 10^{-6} C/m^2 respectively. Find the magnitude of the component of the electric field perpendicular to axis passing through the centers of the two plates at $(10\text{ cm}, 1\text{ mm}, 0)$.

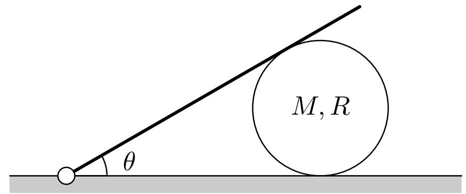
Solution 15: By symmetry, the electric field due to the portion of the plates between $y = 50$ cm and $y = -49.8$ cm will cancel out. We only need to consider the electric field from two 2 mm wide strips of charge, which are small enough to be approximated as wires with charge per unit length $\lambda = \sigma w = \pm 2\times 10^{-9}$ C/m^2 at a distance $y = 50$ cm away. The y-component of the electric field from these wires is then

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\int_{-0.5}^{0.5}\left(\frac{1}{(z^2+0.5^2+0.05^2)^{\frac{3}{2}}} - \frac{1}{(z^2+0.5^2+0.25^2)^{\frac{3}{2}}}\right)\mathrm{d}z$$

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\left(\frac{z}{(0.5^2+0.05^2)\sqrt{z^2+0.5^2+0.05^2}} - \frac{z}{(0.5^2+0.25^2)\sqrt{z^2+0.5^2+0.25^2}}\right)$$

$$E_y = \frac{0.5\lambda}{4\pi\epsilon_0}\left(\frac{1}{(0.5^2+0.05^2)\sqrt{0.5^2+0.5^2+0.05^2}} - \frac{1}{(0.5^2+0.25^2)\sqrt{0.5^2+0.5^2+0.25^2}}\right) = \boxed{11.9 \text{ N/C}}$$

16. Sliding Along A hollow sphere of mass M and radius R is placed under a plank of mass $3M$ and length $2R$. The plank is hinged to the floor, and it initially makes an angle $\theta = \frac{\pi}{3}$ rad to the horizontal. Under the weight of the plank, the sphere starts rolling without slipping across the floor. What is the sphere's initial translational acceleration? Assume the plank is frictionless.



A not-to-scale picture of the sphere-plank setup.

Solution 16: Let N be the normal force acting from the ball on the plank. We can use torque at a distance $x = R\cot\frac{\theta}{2}$ from the hinge (by geometry) to write (using $m = 3M$)

$$mg\frac{l}{2}\cos\theta - NR\cot\frac{\theta}{2} = \frac{ml^2}{3}\alpha$$

You can also find N from

$$N\sin\theta - f = Ma_0$$
$$fR = \frac{2}{3}MR^2\alpha$$
$$a_0 = R\alpha$$

You can relate the acceleration at the point of contact $a_p = \vec{a}_0 + \vec{a}_0 \times \vec{r} = \alpha_0(1-\cos\theta)$. Hence combining all equations gives $a = \frac{9}{32}g \approx \boxed{2.76\ \mathrm{m/s^2}}$.

Alternate: To avoid internal contact forces, we consider Lagrangian mechanics with generalized coordinate θ. The potential energy of the system is

$$U(\theta) = 3MgR\sin\theta + MgR,$$

and so the θ-generalized force is $F_\theta = \frac{\partial V}{\partial\theta}\Big|_{\theta=\frac{\pi}{3}} = \frac{3}{2}MgR$. The kinetic energy of the system is

$$K(\dot\theta) = \frac{1}{2}\left(\frac{1}{3}\cdot 3M(2R)^2\right)\dot\theta^2 + \frac{1}{2}\left(1+\frac{2}{3}\right)Mv^2,$$

where the extra $\frac{2}{3}$ accounts for the rotational kinetic energy of the sphere. Note that the component of velocity normal to the board is $R\dot\theta\sqrt{3}$, so trigonometry gives

$$v = \frac{R\dot\theta\sqrt{3}}{\sin\theta} = 2R\dot\theta.$$

Substituting,

$$K(\dot\theta) = 2MR^2\dot\theta^2 + \frac{10}{3}MR^2\dot\theta^2 = \frac{16}{3}MR^2\dot\theta^2,$$

but this is also equal to $\frac{1}{2}m_\theta\dot\theta^2$, so the θ-generalized mass is $m_\theta = \frac{32}{3}MR^2$. By the Euler-Lagrange equation,

$$\ddot\theta = \frac{F_\theta}{m_\theta} = \frac{9}{64}\frac{g}{R},$$

and we also have $a = 2R\ddot\theta$, so $a = \frac{9}{32}g \approx 2.76\ \mathrm{m/s^2}$.

The following information applies for the next two problems. A space elevator consists of a heavy counterweight placed near geostationary orbit, a thread that connects it to the ground (assume this is massless), and elevators that run on the threads (also massless). The mass of the counterweight is 10^7 kg. Mass is continuously delivered to the counterweight at a rate of 0.001 kg/s. The elevators move upwards at a rate of 20 m/s. Assume there are many elevators, so their discreteness can be neglected. The elevators are massless. The counterweight orbits the Earth.

17. Space Elevator 1 Find the minimum possible displacement radially of the counterweight. Specify the sign.

18. Space Elevator 2 Assuming a radial displacement that is 10 times what you found in the previous part, find the displacement tangentially of the counterweight. Does it lag or lead Earth's motion?

Solution 17: As the orbit is geostationary, we can balance forces to write that

$$\frac{GM_E}{R_{\mathrm{GS}}^2} = \omega^2 R_{\mathrm{GS}} \implies R_{\mathrm{GS}} = \left(\frac{GM_E T^2}{4\pi}\right)^{1/3}$$

The total mass of masses = 1790 kg $\ll 10^6$ kg which means the displacements are small. The total gravity of masses is

$$\int_{R_E}^{R_{Es}} \frac{GM_E}{x^2}(\lambda dx) = \mathrm{GME}\,\lambda\left(\frac{1}{R_E} - \frac{1}{R_{\cos}}\right) = 2650 \text{ N}.$$

where $\lambda = 5 \cdot 10^{-5}$ kg/m. The total centrifugal of masses is

$$\int_{R_E}^{R_E} \frac{4\pi^2}{T^2} \times (\lambda dx) = \frac{2\pi^2}{T^2}\lambda\left(R_{Es}^2 - R_E^2\right) = 230 \text{ V}$$

Hence, the total force is 2420 N. The outwards force is just

$$F_{\mathrm{out}} = \omega^2 aM - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{GM_E}{x^2}\right) aM$$

where a is the horizontal displacement which is much less than R_E, R_{GS}. Hence, this can be rewritten as

$$F_{\mathrm{out}} = \left(\frac{4\pi^2}{T^2} - 2\frac{GM_E}{R_{\mathrm{GS}}^2}\right) aM = 2420 \text{ N}$$

This implies that $a = \boxed{15.21 \text{ km}}$.

Solution 18: The tension at the ground is $2420 \cdot 9 = 21780$ N. The rate of angular momentum delivered to the masses is then

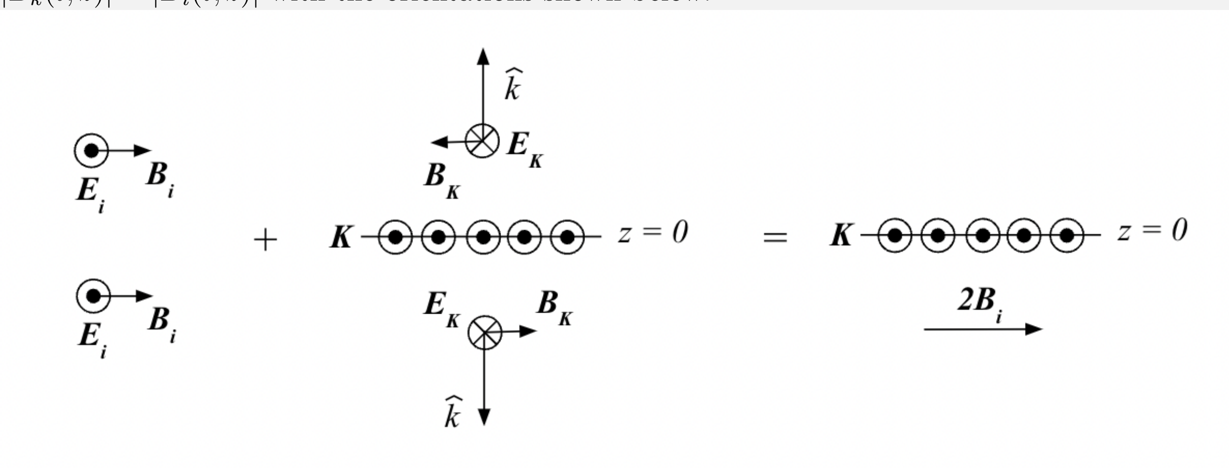
$$\omega(0.001 \text{ kg/s})(R_{\mathrm{GS}}^2 - R_E^2) = 1.27 \cdot 10^8 \text{ kgm}^2/\text{s}^2.$$

This is also the torque acting on the Earth. The horizontal force is 19.9 N which means that $\theta = 19.9/21800$. This tells us that the horizontal displacement is then

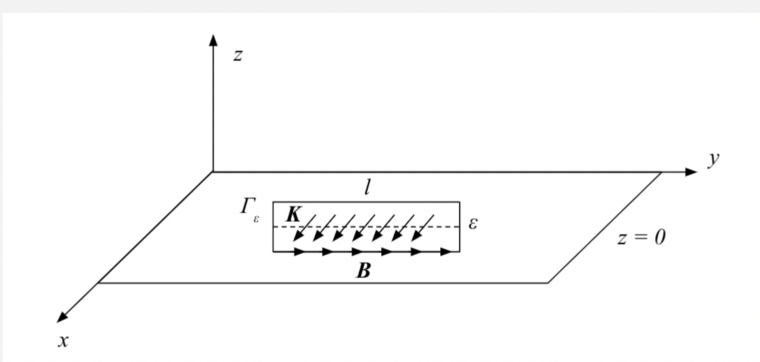
$$b = (R_{\mathrm{GS}} - R_E)\theta = \boxed{32.7 \text{ km}}$$

19. Laser Power Consider a spherical shell of thickness $\delta = 0.5$cm and radius $R = 5$cm made of an Ohmic material with resistivity $\rho = 10^{-7}\Omega$m. A spherical laser source with a tuned frequency of f_0 and intensity $I_0 = 10^5$ W/m^2 is placed at the center of the shell and is turned on. Working in the limit $\delta \ll \frac{c}{f_0} \ll R$, approximate the initial average power dissipated by the shell. Neglect inductance.

Solution 19: Note that the proper treatment of this problem requires more advanced electromagnetism. The following might seem sketchy, but this will do. We first consider the case of a thin plane made of an ideal conductor placed at $z = 0$. A monochromatic electromagnetic wave with an E field: $\vec{E}_i(t, z) = E_0 \cos(kz - \omega t)\hat{x}$ is incident on the plane. The corresponding B field is of course $\vec{B}_i(t, z) = \frac{E_0}{c}\cos(kz - \omega t)\hat{y}$. We expect a current density $\hat{K}(t) = K(t)\hat{x}$ to form in the direction of the E field, as the electric field accelerates the charges on the sheet – $K(t)$ will oscillate with the E field. We know that perfect conductors completely reflect electromagnetic waves, so we expect no EM field in the $z > 0$ region. The total EM field is a combination of the incident wave and the wave generated by the oscillating surface charges, so we are to believe that the latter provides just the right EM wave to cancel out $\vec{B}_i$ and $\vec{E}_i$ in the $z > 0$ region. Consider an instance where the EM waves have an orientation shown in the figure below. It's easy to verify that the magnetic field generated by a uniform sheet of current pointing out of the page is given as shown. Since the electric wave generated by $\vec{K}(t)$ must cancel the $\vec{E}_i$ pointing out of page in $z > 0$, we get a poynting vector (for the wave generated by $\vec{K}$) in the $\hat{z}$ direction in $z > 0$. By symmetry, this means that the poynting vector points in the $-\hat{z}$ direction in $z < 0$. Such simple symmetry arguments fully determine the wave generated by an oscillating sheet of charge: $|\vec{E}_k(t, z)| = |\vec{E}_i(t, z)|$, $|\vec{B}_k(t, z)| = |\vec{B}_i(t, z)|$ with the orientations shown below.



We therefore get that the total magnetic field in the $z < 0$ region is $\vec{B} = 2\vec{B}_i(t, z)$. Let us now take an Ampere loop as shown in the figure below:



We have:

$$T_\epsilon \equiv \oint_{\Gamma_\epsilon} \vec{B}(t, -\epsilon/2) \cdot d\vec{l} = \mu_0 I_{enc} + \frac{1}{c^2}\frac{\partial}{\partial t}\int_{S_\epsilon} \vec{E}(t,z) \cdot d\vec{A}$$

where S_ϵ is the surface bounded by Γ_ϵ. Since $\vec{E} = 0$ everywhere and $I_{enc} = K(t)l$,

$$\lim_{\epsilon \to 0} T_\epsilon = \frac{2E_0 l}{c}\cos(\omega t) = \mu_0 K(t) l$$

Hence, we have $\vec{K}(t) = \frac{2E_0}{\mu_0 c}\cos(\omega t)\hat{x}$. The key find in this long introduction is that an oscillating sheet of current $\vec{K}(t) = K_0 \cos(\omega t)\hat{x}$ generates an EM wave with an electric field $\vec{E}_K(t, 0) = -\frac{\mu_0 c}{2}\vec{K}(t)$ near the sheet (at $z = 0$). We finally return to the original problem.

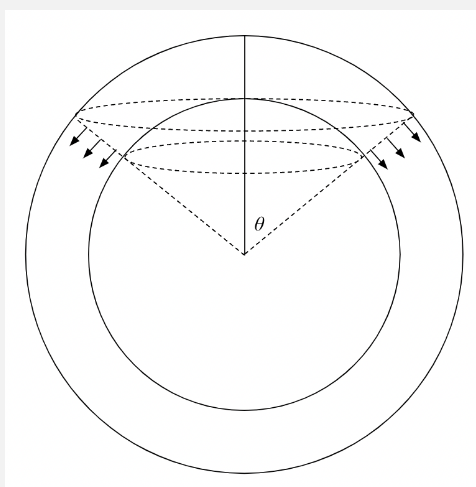
All laser beams have poynting vectors pointing radially outwards, so the E_i field must point in the tangential direction to the sphere at all normal incidence. Say the E_i field points in the $\hat{\theta}$ direction, so that the current in the shell flows from the north pole to the south pole. The condition $\delta \ll \frac{c}{f_0} \ll R$ allows us to neglect attenuation due to skin effects – essentially, the electric field, thus the surface current, is approximately uniform in the shell. The condition $\frac{c}{f_0} \ll R$ allows us to treat the incidence of the laser beams as an EM wave hitting an Ohmic plane of thickness δ, so we consider that problem first. Denote the *total* electric field inside the the plane at time t as $\vec{E}(t)$. The current density is given by Ohm's law: $\vec{J}(t) = \vec{E}(t)/\rho$. The corresponding surface current is of course $\vec{K}(t) = \delta\vec{E}/\rho(t)$. Carefully note the distinction between $\vec{E}$ and $\vec{E}_i$. Since the source wave $\vec{E}_i(t)$ is sinusoidal, we expect $\vec{E}(t)$ to be sinusoidal as well so that $\vec{E}_K(t) = -\frac{\mu_0 c}{2}\vec{K}(t)$ applies. The total electric field inside the sheet is a sum of $\vec{E}_K$ and $\vec{E}_i$, thus:

$$\vec{E}(t) = \vec{E}_i(t) - \frac{\mu_0 c\delta}{2\rho}\vec{E}(t)$$

Rearranging for $\vec{E}(t)$, we get:

$$\vec{E}(t) = \frac{\vec{E}_i(t)}{1 + \mu_0 c\delta/2\rho}$$

On the sphere, $\vec{E}(t)$ points in the $\hat{\theta}$ direction and has the same magnitude everywhere on the sphere. The relevant cross-section will be the lateral surface of a truncated cone, as shown below:



The current through this surface is therefore:

$$I(\theta, t) = \frac{\pi\delta(2R+\delta)|\vec{E}_i(t)|}{\rho + \mu_0 c\delta/2}\sin\theta$$

The power dissipated by a volume generated by $\theta \sim \theta + d\theta$ is

$$dP(t) = I^2(\theta, t)\frac{\rho(R+\delta/2)d\theta}{\pi\delta(2R+\delta)\sin\theta}$$

We integrate this expression from $\theta = 0$ to $\theta = \pi$:

$$P(t) = \frac{\pi\delta\rho(2R+\delta)^2}{(\rho+\mu_0 c\delta/2)^2}|\vec{E}_i(t)|^2$$

Now, the incident electric field has the form $|\vec{E}_i(t)|^2 = \frac{2I_0}{\epsilon_0 c}\cos^2(2\pi f_0 t)$. The average of this function from $t = 0$ to $t = \frac{1}{f_0}$ is 1/2, so our final answer is:

$$\bar{P} = \frac{\pi\delta\rho(2R+\delta)^2}{(\rho+\mu_0 c\delta/2)^2}\frac{I_0}{\epsilon_0 c}$$

which turns out to be $2.39078 \times 10^{-15} W$

20. Twinkle Twinkle. A stable star of radius R has a mass density profile $\rho(r) = \alpha(1 - r/R)$. Here, "stable" means that the star doesn't collapse under its own gravity. If the internal pressure at the core is provided solely by the radiation of photons, calculate the temperature at the core. Assume the star is a perfect black body and treat photons as a classical ideal gas. Use $R = 7 \times 10^5$ km and $\alpha = 3$ g/cm^3. Round your answer to the nearest kilokelvin. We treat photons as a classical gas here to neglect any relativistic effects.

Solution 20: For a star that doesn't collapse in on itself, there must be some source of pressure $p(r)$ that balances out the pressure due to gravity. Define the function:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

inside the star. The gravitational pressure on an infinitesimal shell of radius r and thickness dr is given by:

$$dp_g = -\frac{Gm(r)\rho(r)4\pi r^2 dr}{r^2 4\pi r^2} = -\frac{Gm(r)\rho(r)}{r^2}dr$$

Hence, the pressure source must provide a pressure gradient $\frac{dp}{dr}$ given by:

$$\frac{dp}{dr} = \frac{Gm(r)\rho(r)}{r^2}$$

If $\rho(r) = \alpha(1 - r/R)$, then $m(r) = \frac{1}{3}\pi\alpha r^3(3r/R - 4)$. We need the pressure gradient:

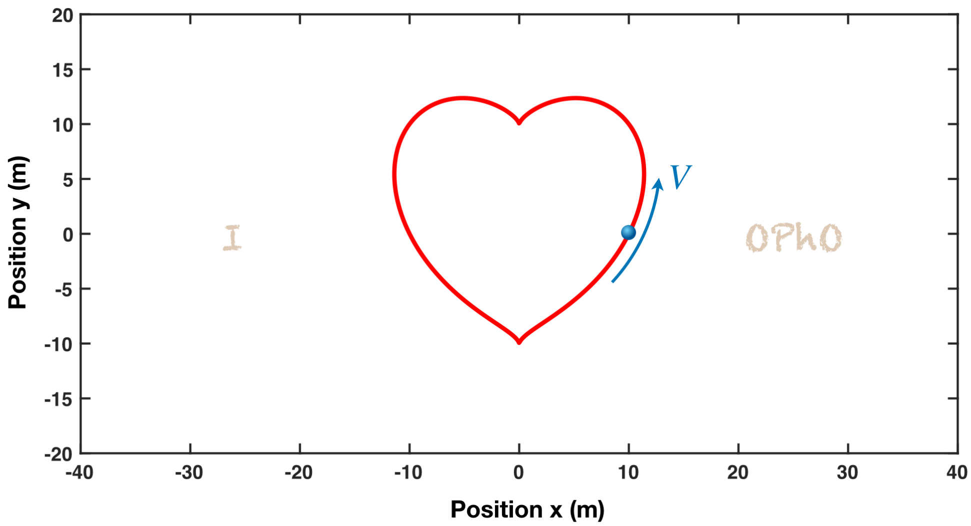
$$\frac{dp}{dr} = \frac{1}{3}\pi G\alpha^2 r(3r/R - 4)(r/R - 1)$$

We integrate this equation from $r = 0$ to R with the obvious boundary condition $p(R) = 0$. We find that $p(0) = \frac{5}{36}\pi G\alpha^2 R^2$. From the Stefan-Boltzmann law, along with elementary kinetic theory, the pressure due to local radiation is given by $\frac{4\sigma}{3c}T_c^4$. Plugging in values gives $T_c = 26718$kK

21. My Heart Will Go On On a flat playground, choose a Cartesian Oxy coordinate system (in unit of meters). A child running at a constant velocity $V = 1\text{m/s}$ around a heart-shaped path satisfies the following order-6 algebraic equation:

$$\left(x^2 + y^2 - L^2\right)^3 - Lx^2y^3 = 0 \ , \ \ L = 10 \ .$$

When the child is at the position $(x, y) = (L, 0)$, what is the magnitude of their acceleration?



Solution 21:
The acceleration can be found from the local geometry of the curves, thus let us study small deviations around the position of interests $(x, y) = (L, 0)$:

$$x = L + \delta_x \quad , \quad y = 0 + \delta_y \quad , \quad |\delta_x|, |\delta_y| \ll L \quad .$$

Consider the 2nd-order approximation in δ_x of δ_y with quadratic coefficients α and β:

$$\delta_y \approx \alpha\delta_x + \frac{\beta}{L}\delta_x^2 \sim \delta_x \quad .$$

To find these coefficients, we look at the algebraic equation of our heart-shape path up to the two lowest-orders of expansions (which are the 3rd and 4th):

$$\begin{aligned} 0 = \left(x^2 + y^2 - L^2\right)^3 - Lx^2y^3 &\approx L^2\left[8L\delta_x^3 + 12\delta_x^4 + 12\delta_x^2\delta_y^2 - 2\delta_x\delta_y^3 - L\delta_y^3 + \mathcal{O}(\delta_x^5)\right] \\ &\approx L^2\left[8L\delta_x^3 + 12\delta_x^4 + 12\alpha^2\delta_x^4 - 2\alpha^3\delta_x^4 - \left(\alpha^3L\delta_x^3 + 3\alpha^2\beta\delta_x^4\right) + \mathcal{O}\left(\delta_x^5\right)\right] \\ &\propto \left(8 - \alpha^3\right)L\delta_x^3 + \left(12 + 12\alpha^2 - 2\alpha^3 - 3\alpha^2\beta\right)\delta_x^4 + \mathcal{O}\left(\delta_x^5\right) \quad . \end{aligned}$$

Thus, α and β can be found by solving:

$$8 - \alpha^3 = 0 \quad , \quad 12 + 12\alpha^2 - 2\alpha^3 - 3\alpha^2\beta = 0 \quad \Longrightarrow \quad \alpha = 2 \ , \ \beta = \frac{11}{3} \quad . \tag{3}$$

We can find the relations between velocities $(\dot{x}, \dot{y}) = \dot{\delta}_x, \dot{\delta}_y)$ and accelerations $(\ddot{x}, \ddot{y}) = \ddot{\delta}_x, \ddot{\delta}_y)$ evaluated at the position $(x, y) = (1, 0) \to (\delta_x, \delta_y) = (0, 0)$ by taking the time-derivatives:

$$\dot{\delta}_y = \alpha\dot{\delta}_x + 2\frac{\beta}{L}\delta_x\dot{\delta}_x = \left(\alpha + 2\frac{\beta}{L}\delta_x\right)\dot{\delta}_x = \alpha\dot{\delta}_x \quad , \tag{4}$$

$$\ddot{\delta}_y = \alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 + 2\frac{\beta}{L}\delta_x\ddot{\delta}_x = \left(\alpha + 2\frac{\beta}{L}\delta_x\right)\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 = \alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2 \quad . \tag{5}$$

For a constant running speed V, we get:

$$V = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \implies \dot{\delta}_x = \left(1+\alpha^2\right)^{-1/2} V \ , \ \dot{\delta}_y = \alpha\left(1+\alpha^2\right)^{-1/2} V \ \ ,$$

which we obtain by applying Eq. (4). Also, the temporal-constraint of constant speed means that the acceleration vector (if non-zero) should be perpendicular to the velocity vector:

$$\frac{d}{dt}V = 0 = \frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \propto \dot{\delta}_x\ddot{\delta}_x + \dot{\delta}_y\ddot{\delta}_y = 0 \implies \ddot{\delta}_x + \alpha\ddot{\delta}_y = 0 \ \ .$$

Using Eq. (5), we can arrive at:

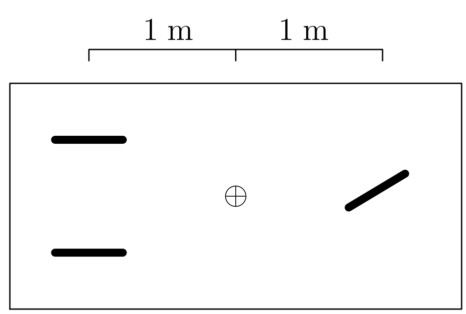
$$\ddot{\delta}_x + \alpha\left(\alpha\ddot{\delta}_x + 2\frac{\beta}{L}\dot{\delta}_x^2\right) = 0 \implies \ddot{\delta}_x = -2\frac{\beta}{L}\alpha\left(1+\alpha^2\right)^{-1}\dot{\delta}_x^2 = -2\beta\alpha\left(1+\alpha^2\right)^{-2}\frac{V^2}{L} \ \ ,$$

$$\ddot{\delta}_y = -\alpha^{-1}\ddot{\delta}_x = 2\beta\left(1+\alpha^2\right)^{-2}\frac{V^2}{L} \ \ .$$

The quadratic coefficients are found in Eq. (3), and given that $V = 1\text{m/s}$, $L = 10\text{m}$, the magnitude of the total acceleration can be calculated:

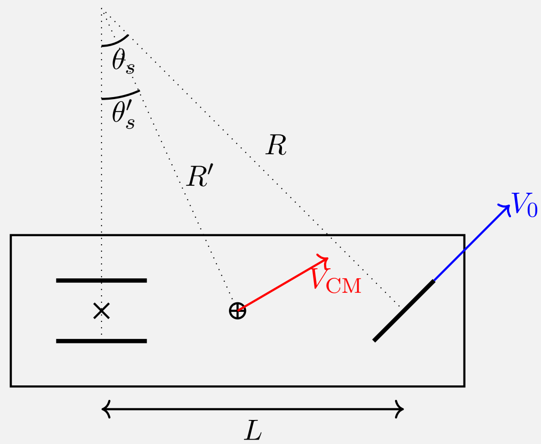
$$a = \left(\ddot{\delta}_x^2 + \ddot{\delta}_y^2\right)^{1/2} = 2\beta\left(1+\alpha^2\right)^{3/2}\frac{V^2}{L} = \frac{22}{15\sqrt{5}}\frac{V^2}{L} \approx \boxed{0.066591\text{m/s}^2} \ \ .$$

22. Tricycle A boy is riding a tricycle across along a sidewalk that is parallel to the x-axis. This tricycle contains three identical wheels with radius 0.5 m. The front wheel is free to rotate while the last two wheels are parallel to each other and to the main body of the tricycle. See the diagram.



The front wheel is rotating at a constant angular speed of $\omega = 3$ rad/s. The child is controlling the tricycle such that the front wheel is making an angle of $\theta(t) = 0.15\sin((0.1 \text{ rad/s})t)$ with the main body of the tricycle. Determine the maximum lateral acceleration in m/s^2. Assume a massless frame. The marked plus sign implies CoM. The degree is in radians.

Solution 22: Assuming the wheels roll without slipping, the cart will instantaneously rotate about a fixed point. This fixed point can be constructed by drawing perpendicular lines from all the wheels and seeing where they intersect. Consider the below diagram,



We have:

$$\sin\theta_s = \frac{L}{R}, \qquad \sin\theta'_s = \frac{L}{2R'}. \tag{6}$$

The angular frequency of every point on the robot is the same $\omega_n = \frac{V_0}{R}$. Therefore,

$$\begin{aligned} V_{CM} &= \omega_n R' && (7)\\ &= \frac{V_0}{R}\frac{L}{2\sin\theta'_s} && (8)\\ &= \frac{V\sin\theta_s}{2\sin\theta'_s}. && (9) \end{aligned}$$

The naive idea here is to say the lateral velocity is $V_{CM}\sin\theta'_s$, but note that the robot could already be rotated. Let the angle of rotation with respect to the horizontal line be α, so we have:

$$V_{CM,lat} = V_{CM}\sin\big(\theta'_s + \alpha\big). \tag{10}$$

Using the sine addition formula, we can simplify this to

$$\begin{aligned} V_{CM,lat} &= V_{CM}\sin\theta'_s\cos\alpha + V_{CM}\cos\theta'_s\sin\alpha && (11)\\ &= \frac{V_0}{2}\sin\theta_s\cos\alpha + \frac{V_0\sin\theta_s}{2\tan\theta'_s}\sin\alpha. && (12) \end{aligned}$$

To simplify this, note that we can write:

$$\frac{\sin\theta_s}{\tan\theta'_s} = 2, \tag{13}$$

where we used the fact that

$$\sin\theta_s \approx \tan\theta_s = 2\tan\theta'_s \approx 2\sin\theta'_s. \tag{14}$$

Then,

$$\dot{y}_{CM} = \frac{V_0}{2}\theta_s + V_0\alpha. \tag{15}$$

At the center of the two back wheels (marked with an X), the speed is

$$V_{back} = R\cos\theta_s\omega_n = V_0\cos\theta_s \approx V_0 \tag{16}$$

so its lateral velocity is

$$\dot{y}_{back} = V_0 \sin\alpha \approx V_0\alpha. \tag{17}$$

Note that the angle α can be written as

$$\alpha \approx \sin\alpha = \frac{y_{CM} - y_{back}}{(L/2)} \implies \dot{\alpha} = \frac{2}{L}\left(\dot{y}_{CM} - \dot{y}_{back}\right) = \frac{V_0}{L}\theta_s. \tag{18}$$

Taking higher derivatives of $\dot{y}_{CM}$, we have

$$\ddot{y}_{CM} = \frac{V_0}{2}\dot{\theta}_s + V_0\dot{\alpha} \tag{19}$$

$$= \frac{V_0}{2}\dot{\theta}_s + \frac{V_0^2}{L}\theta_s. \tag{20}$$

Plugging in $\theta(t) = 0.15\sin(0.1t)$ and $V_0 = 1.5$ m/s, we can optimize $\ddot{y}_{\text{CM}}$ to get 0.16912 m/s.

23. Sonic Fryer In this problem, we consider a simple model for a thermoacoustic device. The device uses heavily amplified sound to provide work for a pump that can then extract heat. Sound waves form standing waves in a tube of radius 0.25 mm that is closed on both sides, and a two-plate stack is inserted in the tube. A temperature gradient forms between the plates of the stack, and the parcel of gas trapped between the plates oscillates sinusoidally between a maximum pressure of 1.03 MPa and a minimum of 0.97 MPa. The gas is argon, with density 1.78 kg/m^3 and adiabatic constant 5/3. The speed of sound is 323 m/s. The heat pump itself operates as follows:

The parcel of gas starts at minimum pressure. The stack plates adiabatically compress the parcel of gas to its maximum pressure, heating the gas to a temperature higher than that of the hotter stack plate. Then, the gas is allowed to isobarically cool to the temperature of the hotter stack plate. Next, the plates adiabatically expand the gas back to its minimum pressure, cooling it to a temperature lower than that of the colder plate. Finally, the gas is allowed to isobarically heat up to the temperature of the colder stack plate.

Find the power at which the thermoacoustic heat pump emits heat.

Solution 23:

The efficiency of the heat engine is $\epsilon = 1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = 0.0237$. The parcel oscillates between pressures P_1 and P_2 sinusoidally with amplitude $P_0 = \frac{P_1 - P_2}{2}$. For a sound wave, the pressure amplitude is $\rho s_0 \omega v$, where s_0 is the position amplitude and v is the speed of sound.
Then the average power with which the sound wave does work on the plates is

$$\langle P\rangle = \frac{1}{2}\rho\omega^2 s_0^2 A v = \frac{P_0^2}{2\rho v}A,$$

where A is the area of each plate. From this, the heat power generated by the engine is $\langle P\rangle/\epsilon = \boxed{6.47 \text{ W.}}$

The following information applies for the next two problems. For your mass spectroscopy practical you are using an apparatus consisting of a solenoid enclosed by a uniformly charged hollow cylinder of charge density $\sigma = 50\ \mu$C/m^2 and radius $r_0 = 7$ cm. There exists an infinitesimal slit of insular material between the cylinder and solenoid to stop any charge transfer. Also, assume that there is no interaction between the solenoid and the cylinder, and that the magnetic field produced by the solenoid can be easily controlled to a value of B_0.

An electron is released from rest at a distance of $R = 10$ cm from the axis. Assume that it is small enough to pass through the cylinder in both directions without exchanging charge. It is observed that the electron reaches a distance R at different points from the axis 7 times before returning to the original position.

24. Isotope Separator 1 Calculate B_0 under the assumption that the path of the electron does not self-intersect with itself.

Solution 24: By Gauss' law, we have

$$\varepsilon_0 \oint \vec{E} \cdot \mathrm{d}\vec{A} = q_{\text{enc}} \implies \varepsilon_0 (2\pi r h) E(r) = (2\pi r_0 h)\sigma \implies E(r) = \frac{\sigma r_0}{\varepsilon_0 r}\hat{r}.$$

We find the work done on the electron from a distance r to R is

$$W = -\int_R^r qE(r)\mathrm{d}r = \int_r^R \frac{q\sigma r_0}{\varepsilon_0 r}\mathrm{d}r = \frac{q\sigma r_0}{\varepsilon_0}\ln\frac{R}{r}.$$

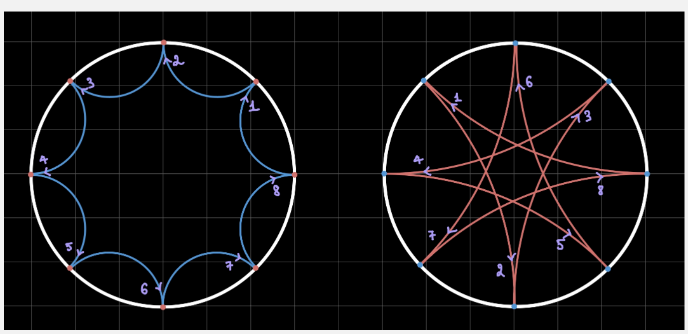
Therefore, by conservation of energy:

$$\frac{1}{2}mv(r)^2 = \frac{q\sigma r_0}{\varepsilon_0}\ln\frac{R}{r} \implies v(r) = \sqrt{\frac{2q\sigma r_0}{m}\ln\frac{R}{r}}.$$

Since the electron is moving in a magnetic field, the radius of the electron follows

$$\frac{mv^2}{a} = qvB \implies a = \frac{mv}{qB} = \sqrt{\frac{2q\sigma r_0}{qB_0^2}\ln\frac{R}{r_0}}$$

This means that $B = \sqrt{\frac{2\sigma m}{q\varepsilon_0 r_0}\ln\frac{R}{r_0}}$. The trajectory of the electron can follow the following paths



As the path is specified to be non-intersecting, we analyze the first path. We can say that the radius of each "circle" the electron travels through is $s = r\tan\frac{\pi}{8} = \frac{mv}{qB}$. Hence,

$$B = \cot\frac{\pi}{8}\sqrt{\frac{2\sigma m}{q\varepsilon_0 r_0}\ln\frac{R}{r_0}}$$

25. Isotope Separator 2 Calculate the time it took for the particle to return to original position. Answer in milliseconds.

Hint: You may find interest in the Gaussian error function:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t.$$

Specific values of the error function can be calculated on desmos.

Solution 25: From the previous part, we have the velocity of the electron is

$$v = \sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0} \ln\left(\frac{R}{r}\right)}$$

Let T_0 represent the time from when the electron is released to when it enters the cylinder. Rewriting as a differential equation, separating variables, and integrating, we have

$$\frac{dr}{dt} = -\sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0} \ln\left(\frac{R}{r}\right)}$$

$$\int_R^{r_0} \frac{dr}{\sqrt{\ln\left(\frac{R}{r}\right)}} = -\sqrt{\frac{2q_e \sigma r_0}{m_e \epsilon_0}} T_0$$

$$T_0 = \sqrt{\frac{\pi m_e \epsilon_0}{2q_e \sigma r_0}} R \,\mathrm{erf}\left(\sqrt{\ln\left(\frac{R}{r_0}\right)}\right)$$

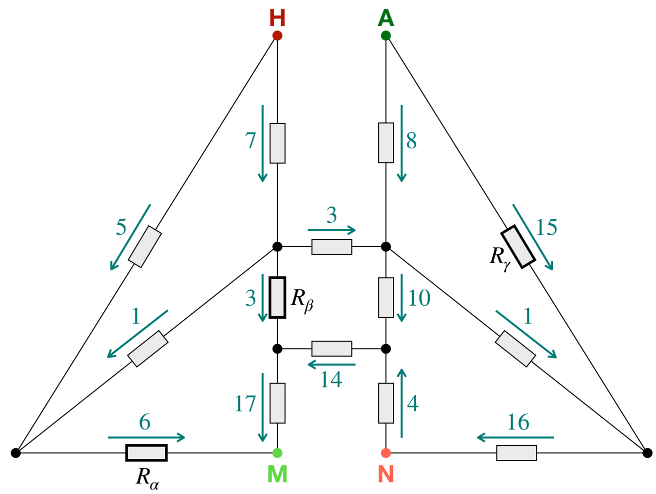
From the previous part, we know that the electron undergoes $\frac{3\pi}{4}$ radians of circular motion with radius of curvature $r_0 \tan\left(\frac{\pi}{8}\right)$ and velocity $v = \sqrt{\frac{2q_e r_0 \sigma}{m_e \epsilon_0} \ln\left(\frac{R}{r_0}\right)}$ Let T_1 be the time from when the electron enters the cylinder to when it exits the cylinder.

$$T_1 = \frac{3\pi}{4} r_0 \tan\left(\frac{\pi}{8}\right) \sqrt{\frac{m_e \epsilon_0}{2q_e r_0 \sigma \ln\left(\frac{R}{r_0}\right)}}$$

The total time is

$$T = 16T_0 + 8T_1 = 5.39 \times 10^{-9} \text{ s} = \boxed{5.39 \times 10^{-6} \text{ milliseconds}}$$

26. Somos El Barco For any circuit network made of batteries and resistors, if we know the voltages of all the batteries and the resistance values of all the resistors, we can calculate all the electrical currents. However, if we know the voltages of all the batteries and all the currents, it is still not enough to uniquely determine the resistance values of all the resistors. Consider a sail-shape circuit network, in which we connect points H and N with a $\mathcal{E}_{\mathrm{HN}} = 10$V battery, points A and M with a $\mathcal{E}_{\mathrm{AM}} = 20$V battery. The electrical currents in this network have directions and magnitudes (in mA) as shown the figure. The possible resistance values of resistors R_α, R_β, R_γ is not a single point (corresponds to an unique solution) but a confined region in the three-dimensional $(R_\alpha, R_\beta, R_\gamma)$-space. Determine the volume of this region (in Ω^3).



The title of this problem means "we are the boat".

Solution 26:
Let us call the nodes on this sail-shape circuit network as in figure A. Since the electrical current always flow toward lower potential, we have the diagram in figure B for the relation between the electrical potential of the nodes (the arrow indicate high-to-low). Use the notation $\mathcal{V}(\bullet)$ to indicate the electrical potential of the point $\bullet$.

The trick for this problem is noticing that the Ohm's laws in our circuit network can always be trivially satisfied for every possible solution following the comparison relationship in figure B, the voltage differences $\mathcal{V}(\mathrm{H}) - \mathcal{V}(\mathrm{N}) = 10\mathrm{V}$, $\mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{M}) = 20\mathrm{V}$ and the equality of in-out currents (in H $5+7$ equals out N $-4+16$, in A $8+15$ equals out M $6+17$) of the batteries. That indeed responsible to why the resistance values of the resistors in this network are not an unique point but rather a region in the 14-dimensional space.

Define:

$$X = \mathcal{V}(\mathrm{P}) - \mathcal{V}(\mathrm{M}) > 0 \ , \ Y = \mathcal{V}(\mathrm{Q}) - \mathcal{V}(\mathrm{U}) > 0 \ , \ Z = \mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{S}) > 0 \ , \ a = \mathcal{V}(\mathrm{U}) - \mathcal{V}(\mathrm{M}) > 0 \ .$$

We want to find the volume in the 3-dimensional subspace of possible $(R_\alpha, R_\beta, R_\gamma)$:

$$\int dR_\alpha dR_\beta dR_\gamma = \frac{\int dX dY dZ}{I_\alpha I_\beta I_\gamma} \ . \tag{21}$$

From figure B and the fixed voltage differences $\mathcal{V}(\mathrm{H}) - \mathcal{V}(\mathrm{N}) = 10\mathrm{V}$, $\mathcal{V}(\mathrm{A}) - \mathcal{V}(\mathrm{M}) = 20\mathrm{V}$, we can write down the inequality equations to specify the boundary of $\{(X, Y, Z)\}$:

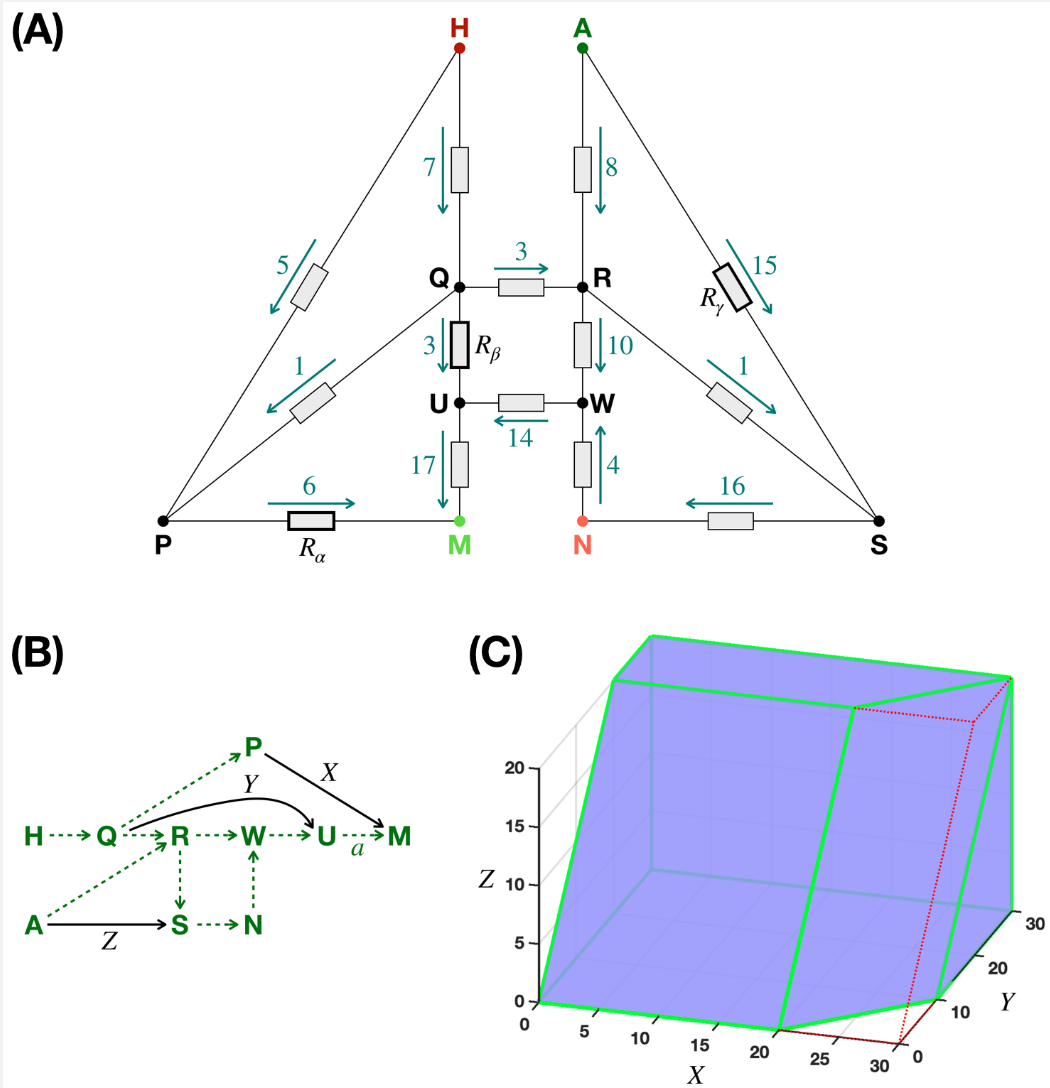
$$a < 20 \ , \ Y < 30 - a \ , \ a < X < a + Y \ , \ Z < \min\left[(20 - a), y\right] \ ,$$

which we can solve to obtain a polygonal-region as shown in figure C (basically, just carving out from the $30 \times 30 \times 20$ rectangle representing maximum bounds of X, Y, Z). Thus:

$$\int dX dY dZ = 30 \times 30 \times 20 - 30 \times \left(\frac{1}{2} \times 20 \times 20\right) - 20 \times \left(\frac{1}{2} \times 10 \times 10\right) = 1.1 \times 10^4 \mathrm{V}^3 \ .$$

Plug this answer back in Eq. (21) and use $I_\alpha = 6\mathrm{mA}$, $I_\beta = 3\mathrm{mA}$, $I_\gamma = 15\mathrm{mA}$:

$$\int dR_\alpha dR_\beta dR_\gamma = \frac{\int dX dY dZ}{I_\alpha I_\beta I_\gamma} = \frac{1.1 \times 10^4 \mathrm{V}^3}{6\mathrm{mA} \times 3\mathrm{mA} \times 15\mathrm{mA}} \approx \boxed{4.0741 \times 10^{10} \Omega^3} \ .$$



* We thank Long T. Nguyen for many useful discussions during the creation of this puzzle.

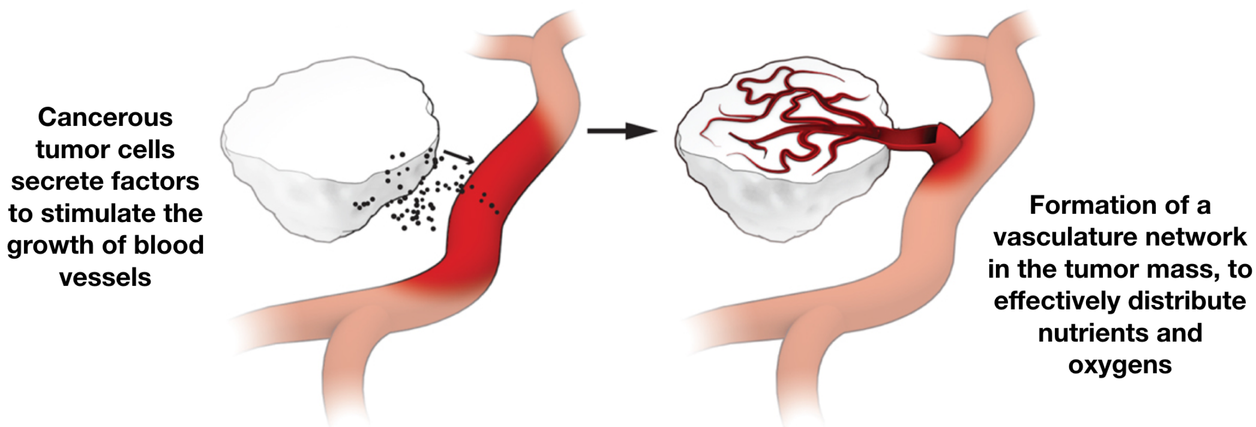
27. The Final Countdown A model of cancer tumor dynamics under a low-dose chemotherapy consists of three non-negative variables (P, Q, R), in which P represents the cancer tumor size, Q represents the (normalized) carrying capability of the tumor vasculature network, and R represents the local (normalized) activity of the immunology system:

$$\frac{\mathrm{d}}{\mathrm{d}t}P = \xi P \ln\frac{Q}{P} - \theta PR - \varphi_1 PC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q = bP - \left(\mu + dP^{2/3}\right) Q - \varphi_2 QC \quad ,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}R = \alpha\left(P - \beta P^2\right) R + \gamma - \delta R + \varphi_3 RC \quad .$$

Here, C is the local concentration of chemotherapeutic agent at the tumor site, which we can assume to follow by a simple pharmacokinetics model:

$$\frac{\mathrm{d}}{\mathrm{d}t}C = -\frac{1}{\tau}C + U \quad ,$$

where U is the rate of chemotherapy drug administrated to the patient body. Let us assume an unchanging rate U, and treat it as another parameter of the model. All other unmentioned symbols are positive constant parameters, which values can be measured and should depends on the particular kinds of cancers and treatments. In total, there are 14 parameters – such a high-degree of complexity is very common in biophysical models. For each set of parameters, there can be many possible stationary states, which can be associated with various levels of malignancy. What is the maximum number of stationary non-zero tumor sizes (including both stable and unstable ones) for a set of parameters in this model? **For this problem, you can only submit your answer once.**



Solution 27:

For a constant drug-rate U, at stationary states the drug-concentration has to be $C = \tau U$. If (P_*, Q_*, R_*) are stationary solutions of this model, then set $C = \tau U$ and all the time-derivatives d/dt... to 0, we obtain:

$$0 = \xi P_* \ln \frac{Q_*}{P_*} - \theta P_* R_* - \varphi_1 \tau U P_* \quad , \tag{22}$$

$$0 = bP_* - \left(\mu - dP_*^{2/3}\right) Q_* - \varphi_2 \tau U Q_* \quad , \tag{23}$$

$$0 = \alpha \left(P_* - \beta P_*^2\right) + \gamma - \delta R_* + \varphi 3 \tau U R_* \quad . \tag{24}$$

From Eq. (22) and Eq. (23), we arrive at the relations:

$$R_* = -\frac{1}{\theta}\left[\xi \frac{P_*}{Q_*} + \varphi_1 \tau U\right] \quad , \quad Q_* = \frac{bP_*}{\mu + \varphi_2 \tau U + dP_*^{2/3}} \quad .$$

Plug these into Eq. (24), we have an equality:

$$\xi \ln\left(\frac{\mu + \varphi_2 \tau U + dP_*^{2/3}}{b}\right) + \varphi_1 \tau U = -\frac{\theta\gamma}{\alpha\beta P_*^2 - \alpha P_* + \delta - \varphi_3 \tau U} \quad .$$

Define the LHS to be $\Phi(P_*)$ and the RHS to be $\Psi(P_*)$. The number of positive solutions P_* for this equation is the number of stationary states that we need to find. Between any two solutions of $\Psi(P_*) = \Phi(P_*)$, there should be a solution of the equation for the derivatives $\Psi'(P_*) = \Phi'(P_*)$, where we use the notation $' = d/dP_*$. We can calculate that:

$$\Phi'(P_*) = \frac{2}{3} \frac{\xi P_*^{-1/3}}{\mu + \varphi_2 \tau U + dP_*^{2/3}} \quad , \quad \Psi'(P_*) = \frac{\theta\gamma\alpha\left(2\beta P_* - 1\right)}{\left(\alpha\beta P_*^2 - \alpha P_* + \delta - \varphi_3 \tau U\right)^2} \quad ,$$

and since $\Phi'(P_*)$ is always positive, therefore the solutions of the derivatives equality can only exist for $P_* > 1/2\beta$. After some algebraic manipulation, $\Psi'(P_*) = \Phi'(P_*)$ becomes:

$$0 = \frac{1}{3}\frac{\xi d}{\theta\gamma}\left[\left(P_* - \frac{1}{2\beta}\right) + \Xi\right]^2 \left(P_* - \frac{1}{2\beta}\right)^{-1} - \left[\left(\mu + \varphi_2\tau U\right) P_*^{1/3} + dP_*\right] \quad , \tag{25}$$

where:

$$\Xi = \frac{4\beta\left(\delta - \varphi_3\tau U\right) - \alpha}{4\alpha\beta^2} \quad .$$

But the RHS of Eq.(25) is a convex function for $P_* > 1/2\beta$, which can be shown by looking at its 2nd-derivative d^2/dP_*^2 in this range:

$$\left.\frac{1}{3}\frac{\xi d}{\theta\gamma}\left[6\left(P_* - \frac{1}{2\beta}\right) + 2\Xi^2\left(P_* - \frac{1}{2\beta}\right)^{-3}\right] + \frac{2}{9}\left(\mu + \varphi_2\tau U\right) P_*^{-5/3}\right|_{P_* > \frac{1}{2\beta}} > 0 \ ,$$

therefore it can have at most two zeros. Thus, back to the original problem $\Phi(P_*) = \Psi(P_*)$, the maximum number of non-zero stationary tumor sizes is $2 + 1 = \boxed{3}$.

28. Magnetic Carts Two carts, each with a mass of 300 g, are fixed to move on a horizontal track. As shown in the figure, the first cart has a strong, tiny permanent magnet of dipole moment $0.5\ \mathrm{A}\cdot\mathrm{m}^2$ attached to it, which is aligned along the axis of the track pointing toward the other cart. On the second cart, a copper tube of radius 7 mm, thickness 0.5 mm, resistivity $1.73\cdot 10^{-8}\Omega$, and length 30 cm is attached. The masses of the magnet and coil are negligible compared to the mass of the carts. At the moment its magnet enters through the right end of the copper tube, the velocity of the first cart is 0.3 m/s and the distance between the two ends of each cart is 50 cm, find the minimum distance achieved between the two ends of the carts in centimeters. While on the track, the carts experience an effective coefficient of static friction (i.e., what it would be as if they did not have wheels) of 0.01. Neglect the self-inductance of the copper tube.

A picture of the two cart setup. The black rectangle represents the magnet while the gold rectangle represents the copper tube.

Hint 1: The magnetic field due to a dipole of moment $\vec{\mu}$, at a position $\vec{r}$ away from the dipole can be written as

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{(3\hat{r}(\vec{r}\cdot\vec{\mu}) - \vec{\mu})}{r^3}\hat{r}$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$.

Hint 2: The following mathematical identity may be useful:

$$\int_{-\infty}^{\infty}\frac{u^2 du}{(1+u^2)^5} = \frac{5\pi}{128}.$$

Solution 28:

We analyze the thin dz of copper which center is at distance z away from the magnet. Notice that the thickness of the copper tube ($w = 0.5mm$) is much smaller than its radius. Thus, we may model the copper tube as an infinitesimally thin current loop of radius $a_0 = a + (w/2)$ (note that $\frac{w}{2a}$ is significant!) with resistance $\rho\frac{2\pi a_0}{wdz}$ where $a = 7mm$. Using the formula for the magnetic field due to the dipole moment, it can be found that the B-field along the current loop is:

$$B = \frac{\mu_0\mu}{4\pi}\frac{(2z^2 - r^2)\hat{z} + 3rz\hat{r}}{(r^2 + z^2)^{5/2}}$$

where $\hat{z}$ is the unit vector parallel to the direction of motion of the magnet and $\hat{r}$ is the unit vector pointing radially outward from the center of the current loop. The flux through the loop is then:

$$\Phi_B = \int_0^{a_0} B_z(2\pi r)dr = \frac{1}{2}\mu_0\mu\frac{a_0^2}{(z^2 + a_0^2)^{3/2}}$$

in which Lenz law can be used to get the current of the loop, using the fact $v = -\frac{dz}{dt}$, where v is the relative velocity of the right cart to the left.

$$I = -\frac{d\Phi_B}{dt}(\rho\frac{2\pi a_0}{wdz})^{-1} = \frac{3\mu_0\mu}{4\pi}\frac{a_0 wvzdz}{\rho(a_0^2 + z^2)^{5/2}}$$

The lorentz force on the current loop due to the magnet then obeys:

$$F = I(2\pi a_0)B_r\hat{z} = \frac{9}{8\pi}\mu_0^2\mu^2\frac{a_0^3 wvz^2dz}{\rho(a_0^2 + z^2)^5}\hat{z}$$

which is repulsive as expected. By Newton's third law, the force on the magnet due to the current in the copper tube is:

$$F = -\frac{9}{8\pi\rho}\mu_0^2\mu^2 a_0^3 wv\int_{-x}^{L-x}\frac{z^2dz}{\rho(a_0^2 + z^2)^5}\hat{z}$$

where x is the depth of the magnet into the copper tube. However, when $x, L - x >> a_0$,

$$\int_{-x}^{L-x}\frac{z^2dz}{\rho(a_0^2 + z^2)^5}\hat{z} \approx \frac{a_0^3}{a_0^{10}}\int_{-\infty}^{\infty}\frac{u^2du}{(1 + u^2)^5} = \frac{5\pi}{128a_0^7}$$

Therefore, for as long as the magnet is sufficiently within the tube, the force is instead:

$$F = -\frac{45}{1024\rho a_0^4}\mu_0^2\mu^2 wv\hat{z} = -\gamma v\hat{z}$$

where $\gamma = 0.1815kg/s$.

The velocity of the second cart starts moving when $F > \mu_s Mg$ with $\mu_s = 0.01$ and $M = 0.3kg$. It turns out that $\frac{0.01Mg}{\gamma v_0} \approx 0.5$, so this static friction is overcome very quickly. Once both carts are in motion, and since there is no kinetic friction, momentum is in fact conserved! As we'll see soon, this results in the carts performing an (infinitely-long) perfectly inelastic collision. In addition, it allows us to use a reduced mass of $M_{red} = M/2$ to calculate the closest distance achieved between the two carts. While $x >> a$ is obeyed:

$$-\gamma v_{rel} = \frac{1}{2}M\frac{dv_{rel}}{dt}$$

giving:

$$v_{rel} = v_0 e^{-\frac{2\gamma}{M}t}$$

in which v_{rel} goes to 0 as t goes to ∞, resulting in a perfectly inelastic collision. Since $v_{rel} = -\frac{dx}{dt}$

$$x_{min} = x_0 - \int_0^\infty v_{rel} dt = x_0 - \frac{mv_0}{2\gamma} = 0.252m$$

We see that $x_{min} >> a$, so while messy stuff occurs when $x \lesssim a_0$, accounting for this can only give rise to an error on the order of a_0. A 5% tolerance was given in this problem so that this would be insignificant.

29. De-Terraforming In the far future, the Earth received an enormous amount of charge as a result of Mad Scientist ecilA's nefarious experiments. Specifically, the total charge on Earth is $Q = 1.0 \times 10^{11}$C. (compare this with the current 5×10^5C).
Estimate the maximum height of a "mountain" on Earth that has a circular base with diameter $w = 1.0$ km, if it has the shape of a spherical sector. You may assume that $h_{\max} \ll w$. The tensile strength of rock is 10 MPa.

Solution 29: You can approximate the mound as a conducting sphere of radius r connecting by a wire to the Earth of radius R. We can therefore say that

$$\sigma_{\text{Mound}} \sim \frac{R}{r}\sigma_{\text{Earth}}$$

as voltages are equal and proportional to Q/R. The electrostatic pressure can be given as $P = \frac{\sigma E}{2}$, where $E/2$ comes from the fact that the section does not contribute a force on itself. This can be rewritten as

$$P = \left(\frac{R}{r}\frac{Q}{4\pi R^2}\right) \cdot \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2} = \frac{Q^2}{32\pi\varepsilon_0 R^3 r}.$$

By using Pythagorean theorem, we can find that $r^2 - (r-h)^2 = w^2$ which means that $h = \frac{w^2}{2r}$. We can hence write the tensile strength as

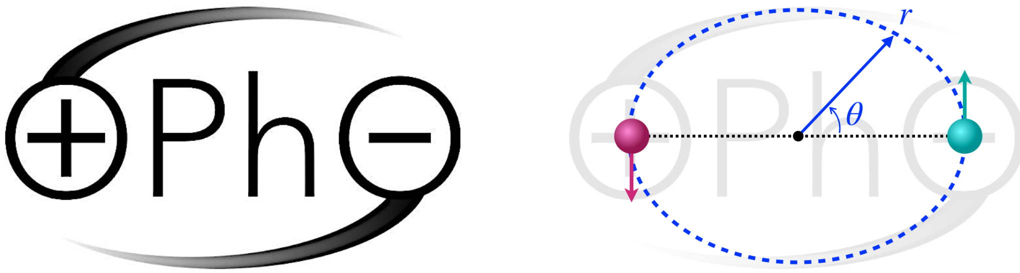
$$Y = \frac{Q^2 h}{16\pi\varepsilon_0 R^3 w^2} \implies h = \frac{16\pi\varepsilon_0 R^3 w^2 Y}{G^2} = \boxed{115 \text{ m}}.$$

Note: As this is an estimate problem, there were many models that could be taken. Some competitors thought that the way electrostatic pressure was calculated was incorrect because it assumes that the force exerted on an element on the mountain is due to the rest of the earth alone. By the same logic, an isolated conducting sphere with a very large surface charge density can exist on its own irrespective of its tensile strength (it would break due to internal free charge repulsions). This argument has validity, but we are considering the fracture surface with the earth while this competitor considered the rocks flying off the mountain itself due to the mountain. It's a grey spot since the problem is an estimate one, so the range was later increased.

30. All Around The World The logo of OPhO describes two objects travelling around their center-of-mass, following the same oval-shape trajectory. For simplicity, we assume these objects are point-like, have identical mass, and interacts via an interacting potential $U(d)$ depends on the distance d between them. Choose the polar coordinates (r, θ) as shown in the figure, where the origin is located at the center of the logo, then the shared trajectory obeys the equation:

$$r(\theta) = \frac{L}{2}\left[1 - \epsilon\cos(2\theta)\right]^{-(1+\gamma)} ,$$

in which we consider $\epsilon = 0.12$ and $\gamma = 0.05$. Here the smallest and largest separation between the objects are d_{min} and d_{max}. Since the interacting potential $U(d)$ is defined up to a constant, let us pick $U(L) = 0$. Find the ratio $U(d_{\text{min}})/U(d_{\text{max}})$.



Solution 30:
Define the dimensionless inverse-radius $u = \frac{L}{d}$ and the angular-derivative $' = \frac{d}{d\theta}$, then for any central interacting potential we have Binet equation (can be derived directly from the equations of motion in polar coordinates):

$$\frac{d}{du}U(u) \propto -\left(u'' + u\right) \quad . \tag{26}$$

For the given oval-shape trajectory:

$$d = L\left[1 - \epsilon\cos(2\theta)\right]^{-(1+\gamma)} \implies u = (1 - \epsilon\cos\phi)^{(1+\gamma)} \ , \ \cos\phi = \epsilon^{-1}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right] \quad , \tag{27}$$

where $\phi = 2\theta$. The angular 2nd-derivative of the dimensionless inverse-radius u can be calculated:

$$\begin{aligned}
u'' = 4\frac{d^2}{d\phi^2}u &= 4\epsilon(1+\gamma)\left[\frac{\gamma\epsilon\left(1 - \cos^2\phi\right)}{1 - \epsilon\cos\phi} + \cos\phi\right](1 - \epsilon\cos\phi)^{\gamma} \\
&= 4\epsilon(1+\gamma)\left\{\gamma\epsilon\left(1 - \epsilon^{-2}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right]^2\right)u^{-\left(\frac{1}{1+\gamma}\right)} + \epsilon^{-1}\left[1 - u^{\left(\frac{1}{1+\gamma}\right)}\right]\right\}u^{\left(\frac{\gamma}{1+\gamma}\right)} \\
&= 4(1+\gamma)\left\{\gamma\epsilon^2 u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} - \gamma u^{-\left(\frac{1-\gamma}{1+\gamma}\right)}\left[1 - 2u^{\left(\frac{1}{1+\gamma}\right)} + u^{\left(\frac{2}{1+\gamma}\right)}\right] + \left[u^{\left(\frac{\gamma}{1+\gamma}\right)} - u\right]\right\} \\
&= -4\gamma(1+\gamma)(1-\epsilon^2)u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} + 4(1+\gamma)(1+2\gamma)u^{\left(\frac{\gamma}{1+\gamma}\right)} - 4(1+\gamma)^2 u \quad .
\end{aligned}$$

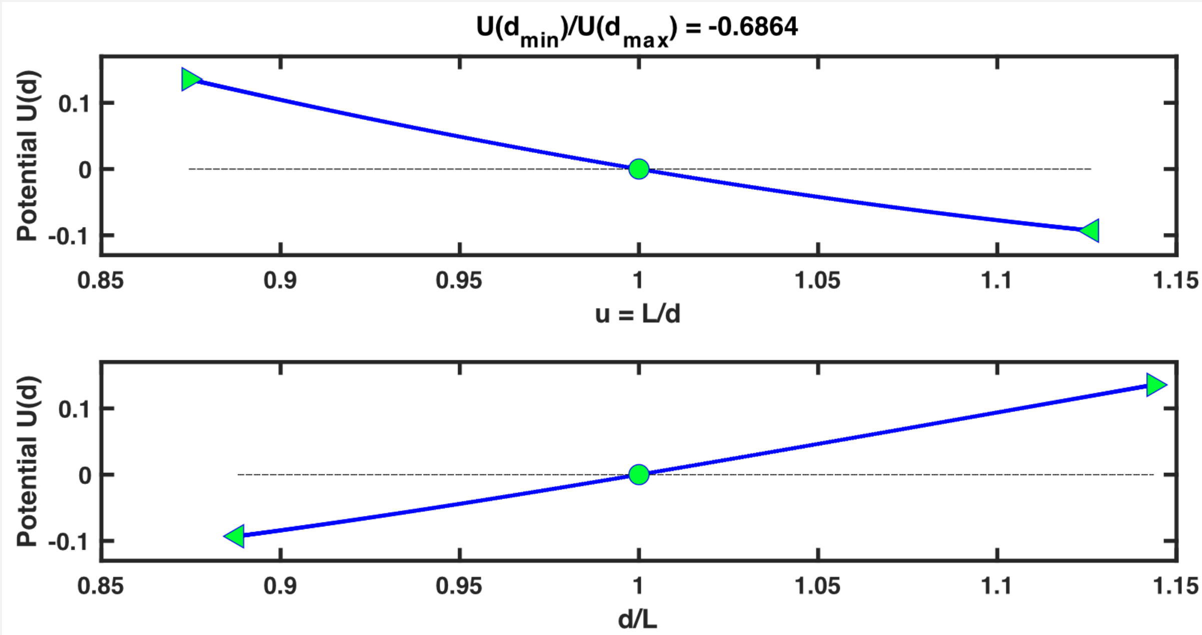
Plug this back into Eq. (26), we can arrive at:

$$\begin{aligned}
&\frac{d}{du}U(u) \propto \left[4(1+\gamma)^2 - 1\right]u + 4\gamma(1+\gamma)(1-\epsilon^2)u^{-\left(\frac{1-\gamma}{1+\gamma}\right)} - 4(1+\gamma)(1+2\gamma)u^{\left(\frac{\gamma}{1+\gamma}\right)} \\
\implies \quad &U(u) \propto \left(2 - \frac{1}{2(1+\gamma)^2}\right)u^2 + 2(1-\epsilon^2)u^{\left(\frac{2\gamma}{1+\gamma}\right)} - 4u^{\left(\frac{1+2\gamma}{1+\gamma}\right)} + W \quad ,
\end{aligned} \tag{28}$$

where W is a constant so that we can assign $U(u=1) = 0$. For $\epsilon = 0.12$ and $\gamma = 0.05$, we get $W = 0.48231$. Following that, we use Eq. (27), the maximum distance corresponds to $\theta = 0$ and thus $u_{\triangleright} = 0.87439$, the minimum distance corresponds to $\theta = \pi/2$ and thus $u_{\triangleleft} = 1.1264$. The ratio of interests, therefore, can be found directly:

$$\frac{U(u = u_{\triangleleft})}{U(u = u_{\triangleright})} \approx \boxed{-0.6864} \quad .$$

We plot the interacting potential, up to a pre-factor as given in Eq. (28), in the figure below.



31. Electrostatic Pendulum 1 Follin is investigating the electrostatic pendulum. His apparatus consists of an insulating Styrofoam ball with a mass of 14 mg and radius $r = 0.5$ cm suspended on a uniform electrically-insulating string of length 1 m and mass per unit length density of $1.1 \cdot 10^{-5}$ kg/m between two large metal plates separated by a distance 17 cm with a voltage drop of $10kV$ between them, such that when the ball is in equilibrium, its center of mass is exactly equidistant to the two plates. Neglect the possibility of electrical discharge throughout the next two problems.

Follin then gives the ball a charge 0.15 nC. Assuming that the charge is distributed evenly across the surface of the ball, find the subsequent horizontal deflection of the pendulum bob's center of mass from its hanging point at equilibrium.

32. Electrostatic Pendulum 2 Hoping to get a larger deflection, Follin replaces the insulating Styrofoam ball with a conducting pith ball of mass 250 mg and 2 cm and daisy chains 4 additional 10 kV High Voltage Power Supplies to increase the voltage drop across the plates to 50 kV. Leaving the plate separation and the string unchanged, he repeats the same experiment as before, but forgets to measure the charge on the ball. Nonetheless, once the ball reaches equilibrium, he measures the deflection from the hanging point to be 5.6 cm. Find the charge on the ball.

Solution 31: The force on the Styrofoam ball due to the electric field from the two plates is $F = \frac{QV}{d}$. Since the plates are conducting, the ball also experiences an attraction toward the closer plate. This force can be neglected because it is much smaller than $\frac{QV}{d}$ (an assumption that will be justified at the end of the solution). The mass of the string, however, needs to be considered. Consider the forces on an infinitesimal segment of the string. The horizontal component of the tension must balance out the electric force on the ball and the vertical component must balance out the weight of everything below it (string and ball). This gives us

$$\frac{dx}{dh} = \frac{F}{mg + \lambda gh}$$

where h is the height above the ball, x is the horizontal displacement from equilibrium, F is the electrostatic force, λ is the mass density of the string, and m is the mass of the ball. Separating variables, we have

$$x = F \int_0^L \frac{dh}{mg + \lambda gh}$$

where L is the length of the string. Integrating, we get

$$x = \frac{F}{\lambda g} \ln\left(\frac{m + \lambda L}{m}\right) = \boxed{0.0475 \text{ m}}$$

Now, let's justify our assumption that the attraction of the ball towards the plates is negligible. Using the method of image charges, the force due to the closer plate is equivalent to a charge $-Q$ a distance of $d - 2x$ away. Infinitely many image charges exist due to the other plate but they have a smaller effect so they will not be considered. The force due to the image charge is

$$F' = \frac{Q^2}{4\pi\epsilon_0 (d - 2x)^2} = 3.59 \times 10^{-8} \text{ N}$$

which is indeed much less than $F = \frac{QV}{d} = 8.82 \times 10^{-6}$ N

Solution 32: When a conducting ball is placed in a uniform electric field, the charges separate so that the sphere becomes a dipole with electric dipole moment

$$p = 4\pi\epsilon_0 r^3 E_0$$

Let Q be the charge on the ball. The ball can be approximated as a point dipole with dipole moment p and a point charge of magnitude Q at the center of the ball. Since the plates are conducting and must be equipotential, the charge and dipole will experience an attractive force to the plates. Using the method of image charges, the force caused by the closer plate is equivalent to that of a point charge $-Q$ and a dipole moment p at a distance $d - 2x$ from the center of the ball, where $d = 17$ cm is the separation between the plates and $x = 5.6$ cm is the deflection of the ball from the hanging point. The forces from the farther plate are negligible compared to the force due to the electric field, QE_0. The force between a point charge Q and a point dipole moment p a distance r apart is

$$F = \frac{pQ}{2\pi\epsilon_0 r^3}$$

The force between two dipoles with dipole moments of magnitude p in the same direction as the vector from one dipole to the other is

$$F = \frac{3p^2}{2\pi\epsilon_0 r^4}$$

Adding up the charge-image charge, charge-image dipole, dipole-image charge, and dipole-image dipole interactions, we get that the total electrostatic force on the ball is

$$F = QE_0 + \frac{Q^2}{4\pi\epsilon_0 (d - 2x)^2} + \frac{pQ}{\pi\epsilon_0 (d - 2x)^3} + \frac{3p^2}{2\pi\epsilon_0 (d - 2x)^4}$$

From the previous problem, we know that the displacement x of the ball from equilibrium as a function of the electrostatic force F is

$$x = \frac{F}{lg} \ln\left(\frac{m + l}{m}\right)$$

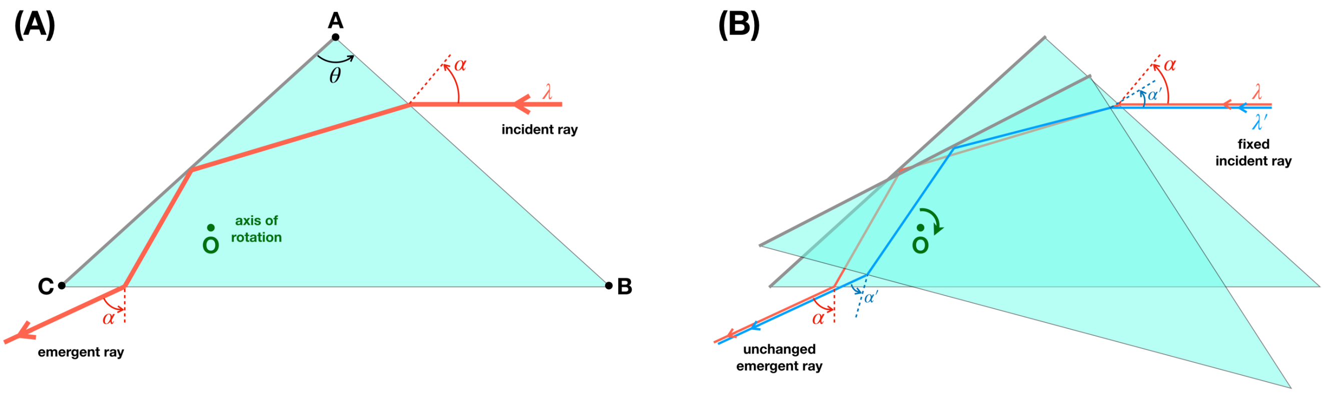
Rearranging this as a quadratic in Q and solving, we have

$$\frac{1}{4\pi\epsilon_0 (d-2x)^2}Q^2 + \left(\frac{p}{\pi\epsilon_0 (d-2x)^3} + E_0\right)Q + \left(\frac{3p^2}{2\pi\epsilon_0 (d-2x)^4} - \frac{lgx}{\ln\left(\frac{m+l}{m}\right)}\right) = 0$$

$$Q = \frac{-\left(\frac{p}{\pi\epsilon_0(d-2x)^3} + E_0\right) + \sqrt{\left(\frac{p}{\pi\epsilon_0(d-2x)^3} + E_0\right)^2 - 4\left(\frac{1}{4\pi\epsilon_0(d-2x)^2}\right)\left(\frac{3p^2}{2\pi\epsilon_0(d-2x)^4} - \frac{lgx}{\ln\left(\frac{m+l}{m}\right)}\right)}}{2\frac{1}{4\pi\epsilon_0(d-2x)^2}}.$$

Plugging gives $Q = \boxed{4.48 \times 10^{-10}\ \text{C}}$.

The following information applies for the next two problems. Consider a uniform isosceles triangle prism ABC, with the apex angle $\theta = 110°$ at vertex A. One of the sides, AC, is coated with silver, allowing it to function as a mirror. When a monochrome light-ray of wavelength λ approaches side AB at an angle of incidence α, it first refracts, then reaches side AC, reflects, and continues to base BC. After another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence (see Fig. A).



33. Man In The Mirror 1 What is the relative refractive index of the prism for that particular wavelength λ with respect to the outside environment, given that $\alpha = 70°$.

34. Man In The Mirror 2 Consider keeping the incident ray fixed while changing the monochrome color to a different wavelength λ', so that by rotating the prism around an axis of rotation O it can follows the above description (approaches side AB at an angle of incidence α', refracts, then reaches side AC, reflects, and continues to base BC, then another refraction, the ray eventually exits the prism at the angle of emergence which is also equal to the angle of incidence). We observe then the emergent ray remains unchanged for any value of λ' (see Fig. B). Find the maximum possible length-ratio between the distance from the axis to the one of the vertices $\max(\mathrm{OA}, \mathrm{OB}, \mathrm{OC})$ and the base BC.

Solution 33: The light-path refracts on side AB at point M, reflects on side AC at point N and refracts on base BC at point P (see Fig. A). Define the angle of refraction inside the prism to be β, then from Snell's law:

$$\sin\alpha = n\sin\beta\ . \tag{29}$$

From the law of reflection and the 180^o-sum of three interior angles inside any triangles:

$$\widehat{\text{MNA}} = 180^o - \widehat{\text{NAM}} - \widehat{\text{AMN}} = 180^o - \theta - (90^o - \beta)$$
$$= \widehat{\text{PNC}} = 180^o - \widehat{\text{NCP}} - \widehat{\text{CPN}} = 180^o - \left(\frac{180^o - \theta}{2}\right) - (90^o + \beta) \ ,$$

we obtain the refraction angle β to be:

$$\beta = \frac{3\theta - 180^o}{4} \ .$$

Plug this finding into Eq. (29), we get the relative refraction index of the prism with respect to the outside environment:

$$n = \frac{\sin\alpha}{\sin\beta} = \frac{\sin\alpha}{\sin\left(\frac{3\theta - 180^o}{4}\right)}\Bigg|_{\alpha=70^o,\theta=110^o} \approx \boxed{1.5436} \ .$$

Solution 34: Another wavelength λ' corresponds to a new relative refractive index n' and therefore a new angle angle of incidence and emergence α'. Say, the new light-path after rotating the prism around the O axis is through a refraction at point M' on side AB, a reflection at point N' on side AC, and a refraction P' on base BC. Choose a parallelogram Bac coordinates where Ba pointing in the same direction as vector $\overrightarrow{\text{BA}}$ and Bc pointing in the same direction as vector $\overrightarrow{\text{BC}}$ (see Fig. B). Define the angle:

$$\gamma = \widehat{\text{ABC}} = \widehat{\text{BCA}} = \frac{180^o - \theta}{2} \ .$$

Due to symmetry (equal angles of incidence and emergence), the refraction angle inside the prism is always $\beta = 90^o - 3\gamma/2$ and the deviation angle between the incident and emergent rays is always γ, regardless of α'. The vector $\overrightarrow{\text{P'N'}}$ made an angle $\gamma - 90^o - \beta = 5\gamma/2 - 180^o$ with Ba and therefore the line equation of M'N' in Bac is given by:

$$\frac{c - c_{\text{P'}}}{a - a_{\text{P'}}} = \frac{c - c_{\text{P'}}}{a} = \frac{\sin\left(5\gamma/2 - 180^o\right)}{\sin\left[\gamma - \left(5\gamma/2 - 180^o\right)\right]} = -\frac{\sin\left(5\gamma/2\right)}{\sin\left(3\gamma/2\right)} \ , \tag{30}$$

where here we denote $(a_{\text{X}}, c_{\text{X}})$ as the position of point X in the parallelogram Bac coordinates. Similarly, the line equation of CA (which vector $\overrightarrow{\text{CA}}$ makes an angle $2\gamma - 180^o$ with Ba) Bac is given by:

$$\frac{c - c_{\text{C}}}{a - a_{\text{C}}} = \frac{c - c_{\text{C}}}{a} = \frac{\sin\left(2\gamma - 180^o\right)}{\sin\left[\gamma - \left(2\gamma - 180^o\right)\right]} = -\frac{\sin\left(2\gamma\right)}{\sin\left(\gamma\right)} \ . \tag{31}$$

Since N' is a point on line CA, the position N' $(a_{\text{N'}}, c_{\text{N'}})$ must satisfy both Eq. (30) and Eq. (31), which we can solve to obtain:

$$a_{\text{N'}} = \frac{(c_{\text{C}} - c_{\text{P'}})\sin\left(3\gamma/2\right)}{\sin\left(\gamma/2\right)} \ , \ c_{\text{N'}} = \frac{c_{\text{P'}}\sin\left(3\gamma/2\right)\sin\left(2\gamma\right) - c_{\text{C}}\sin\left(\gamma\right)\sin\left(5\gamma/2\right)}{\sin\left(\gamma/2\right)\sin\left(\gamma\right)} \ . \tag{32}$$

Next, we consider the line M'N'. Vector $\overrightarrow{\text{N'M'}}$ makes an angle $-90^o - \beta = 3\gamma/2 - 180^o$ with Ba, thus, in Bac the line N'M' should obeys the equation:

$$\frac{c - c_{\text{N'}}}{a - a_{\text{N'}}} = \frac{\sin\left(3\gamma/2 - 180^o\right)}{\sin\left[\gamma - \left(3\gamma/2 - 180^o\right)\right]} = -\frac{\sin\left(3\gamma/2\right)}{\sin\left(\gamma/2\right)} \ .$$

Since point M' in also on side AB, therefore $c_{\mathrm{M'}} = 0$ and this leads to:

$$a_{\mathrm{M'}} = \frac{a_{\mathrm{N'}} \sin(3\gamma/2) + c_{\mathrm{N'}} \sin(\gamma/2)}{\sin(3\gamma/2)} = \frac{2c_{\mathrm{C}} \sin\gamma \cos(\gamma/2) - c_{\mathrm{P'}} \sin(3\gamma/2)}{\sin(3\gamma/2)} \, , \tag{33}$$

where for the final algebra manipulation we use Eq. (32).

Now let us choose another parallelogram Oxy coordinates where Ox pointing in the same direction as the incident ray and Oy pointing in the same direction as the emergent ray (see Fig. B). Note that $\widehat{x\mathrm{O}y} = \widehat{a\mathrm{B}c} = \gamma$, and the angle between Ba and Ox is $\alpha' - 90^o$. The coordinate transformation $(a, c) \to y$ due to pure translation and rotation:

$$y - y_{\mathrm{B}} = \frac{a \sin(\alpha' - 90^o) + c \sin[\gamma + (\alpha' - 90^o)]}{\sin\gamma} = -\frac{a \cos\alpha' + c \cos(\gamma + \alpha')}{\sin\gamma} \, , \tag{34}$$

where here we denote $(x_{\mathrm{X}}, y_{\mathrm{X}})$ as the position of point X in the parallelogram Bxy coordinates. Hence the y-coordinates of point M' is given by:

$$y_{\mathrm{M'}} = y_{\mathrm{B}} - \frac{a_{\mathrm{M'}} \cos\alpha'}{\sin\gamma} = y_{\mathrm{B}} - \frac{[2c_{\mathrm{C}} \sin\gamma \cos(\gamma/2) - c_{\mathrm{P'}} \sin(3\gamma/2)] \cos\alpha'}{\sin(3\gamma/2) \sin\gamma} \, , \tag{35}$$

in which we utilize the result found in Eq. (33). Similarly, for $(a, c) \to x$:

$$x - x_{\mathrm{B}} = \frac{a \cos(\gamma - \alpha') + c \cos\alpha'}{\sin\gamma} \, , \tag{36}$$

consider point P' on the x-axis we also arrive at the following relationship with Eq. (36):

$$x_{\mathrm{P'}} = x_{\mathrm{B}} - \frac{c_{\mathrm{P'}} \cos\alpha'}{\sin\gamma} \, ,$$

and express $(x_{\mathrm{B}}, y_{\mathrm{B}})$ in terms of $(a_{\mathrm{O}}, c_{\mathrm{O}})$ with Eq. (36) and Eq. (34):

$$\begin{aligned} x_{\mathrm{O}} - x_{\mathrm{B}} &= \frac{a_{\mathrm{O}} \cos(\gamma - \alpha') + c_{\mathrm{O}} \cos\alpha'}{\sin\gamma} \quad \Longrightarrow \quad x_{\mathrm{B}} = -\frac{a_{\mathrm{O}} \cos(\gamma - \alpha') + c_{\mathrm{O}} \cos\alpha'}{\sin\gamma} \, , \\ y_{\mathrm{O}} - y_{\mathrm{B}} &= -\frac{a_{\mathrm{O}} \cos\alpha' + c_{\mathrm{O}} \cos(\gamma + \alpha')}{\sin\gamma} \quad \Longrightarrow \quad y_{\mathrm{B}} = \frac{a_{\mathrm{O}} \cos\alpha' + c_{\mathrm{O}} \cos(\gamma + \alpha')}{\sin\gamma} \, . \end{aligned}$$

Using these to replace $c_{\mathrm{P'}}$ and $(x_{\mathrm{B}}, y_{\mathrm{B}})$ in Eq. (35) and rearrange the terms, we end up with:

$$x_{\mathrm{P'}} = y_{\mathrm{M'}} - (a_{\mathrm{O}} - c_{\mathrm{O}}) \sin\alpha' - \left[(a_{\mathrm{O}} + c_{\mathrm{O}}) - \frac{c_{\mathrm{C}} \sin\gamma}{\cos(\gamma/2) \sin(3\gamma/2)}\right] \cot(\gamma/2) \cos\alpha' \, , \tag{37}$$

where the emergent ray position $x_{\mathrm{P'}}$ is a function of the incident ray position $x_{\mathrm{P'}}$, the angle of incidence/emergence α', and constants of prism rotation: γ, $(a_{\mathrm{O}}, a_{\mathrm{O}})$.

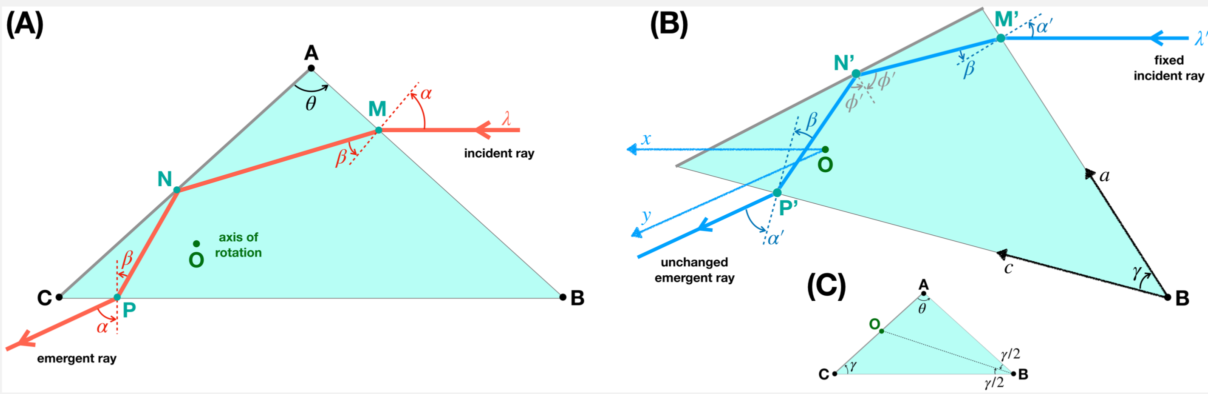
If we keep the incident ray fixed $y_{\mathrm{M'}} =$ const and rotate the prism after changing the monochromatic wavelength λ' (varying α'), to have the emergent ray stays the same $x_{\mathrm{P'}} =$ const, followed from Eq. (37) we need:

$$\begin{aligned} 0 = a_{\mathrm{O}} - c_{\mathrm{O}} &= (a_{\mathrm{O}} + c_{\mathrm{O}}) - \frac{c_{\mathrm{C}} \sin\gamma}{\cos(\gamma/2) \sin(3\gamma/2)} \\ \Longrightarrow \quad a_{\mathrm{O}} = c_{\mathrm{O}} &= \frac{c_{\mathrm{C}} \sin\gamma}{2\cos(\gamma/2) \sin(3\gamma/2)} \, . \end{aligned} \tag{38}$$

We have found the position of the rotational axis O, independence of the position where the incident ray hits the side AB! The constraint $a_O = c_O$ means it is on the line bisecting the angle between the incidence and the emergence faces of the prism $\widehat{\mathrm{ABC}}$. We also note that the solution found in Eq. (38) satisfy the line equation of CA which is given in Eq. (31).

In summary, the rotational axis O is the intersection between the bisector of angle $\widehat{\mathrm{ABC}}$ and line CA (see Fig. C). After some geometrical analysis, we get the result:

$$\frac{\max\left(\mathrm{OA},\mathrm{OB},\mathrm{OC}\right)}{\mathrm{BC}} = \frac{\mathrm{OB}}{\mathrm{BC}} = \frac{\sin\widehat{\mathrm{OCB}}}{\sin\widehat{\mathrm{BOC}}} = \frac{\sin\gamma}{\sin\left(180^o - \frac{3}{2}\gamma\right)} = \frac{\sin\left(\frac{180^o-\theta}{2}\right)}{\sin\left(\frac{3\theta+180^o}{4}\right)}\Bigg|_{\theta=110^o} \approx \boxed{0.72300}\ .$$



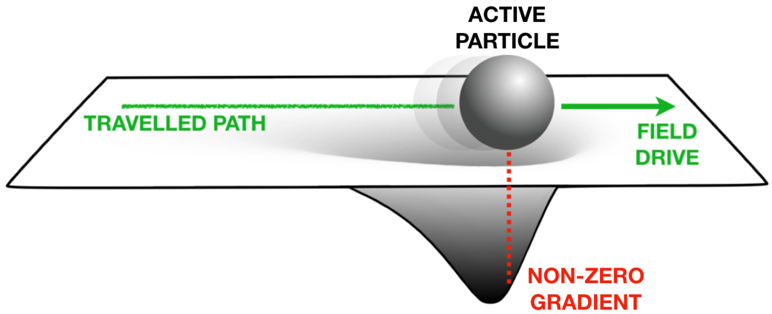
35. Funiculì, Funiculà Field-drive is a locomotion mechanism that is analogous to general relativistic warp-drive. In this mechanism, an active particle continuously climbs up the field-gradient generated by its own influence on the environment so that the particle can bootstrap itself into a constant non-zero velocity motion. Consider a field-drive in one-dimensional (the Ox axis) environment, where the position of the particle at time t is given by $X(t)$ and its instantaneous velocity follows from:

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \kappa\frac{\partial}{\partial x}R(x,t)\Big|_{x=X(t)}\ ,$$

in which κ is called the guiding coefficient and $R(x,t)$ is the field-value in this space. Note that, the operation $...|_{x=X(t)}$ means you have to calculate the part in ... first, then replace x with $X(t)$. For a biological example, the active particle can be a cell, the field can be the nutrient concentration, and the strategy of climbing up the gradient can be chemotaxis. The cell consumes the nutrient and also responses to the local nutrient concentration, biasing its movement toward the direction where the concentration increases the most. If the nutrient is not diffusive and always recovers locally (e.g. a surface secretion) to the value which we defined to be 0, then its dynamics can usually be approximated by:

$$\frac{\partial}{\partial t}R(x,t) = -\frac{1}{\tau}R(x,t) - \gamma\exp\left\{-\frac{[x-X(t)]^2}{2\lambda^2}\right\}\ ,$$

where τ is the timescale of recovery, γ is the consumption, and λ is the characteristic radius of influence. Before we inoculate the cell into the environment, $R = 0$ everywhere at any time. What is the smallest guiding coefficient κ (in $\mu\mathrm{m}^2/\mathrm{s}$) for field-drive to emerge, if the parameters are $\tau = 50\mathrm{s}$, $\gamma = 1\mathrm{s}^{-1}$, and $\lambda = 10\mu\mathrm{m}$.



The title of this problem means "funicular up, funicular down".

Solution 35:
Assume that we inoculate the cell into the environment at position $x = 0$ and $t = 0$. The field dynamics at $t > 0$ can be rewritten as:

$$\begin{aligned} \frac{\partial}{\partial t}R(x,t) + \frac{1}{\tau}R(x,t) = \exp\left(-\frac{t}{\tau}\right)\partial_t\left[\exp\left(+\frac{t}{\tau}\right)R(x,t)\right] &= -\gamma\exp\left\{-\frac{[x-X(t)]^2}{2\lambda^2}\right\} \\ \Longrightarrow \quad \exp\left(+\frac{t}{\tau}\right)R(x,t) &= \int_0^t dt'\exp\left(+\frac{t'}{\tau}\right)\left(-\gamma\exp\left\{-\frac{[x-X(t')]^2}{2\lambda^2}\right\}\right) \\ \Longrightarrow \quad R(x,t) &= -\gamma\int_0^t dt'\exp\left\{-\frac{t-t'}{\tau}-\frac{[x-X(t')]^2}{2\lambda^2}\right\} \ . \end{aligned} \tag{39}$$

If the cell can field-drive at a constant velocity $W > 0$, then after a very long time $t \to +\infty$ we expect the cell will be in a steady-state, moving at this velocity. For consistency, this field-drive velocity W should related to the field gradient evaluated at $x = X(t)$ such that:

$$W = \kappa\partial_x R(x,t)\Big|_{x=X(t)} \ . \tag{40}$$

From Eq. (39) we obtain:

$$\begin{aligned} W &= \kappa\partial_x\left(-\gamma\int_0^t dt'\exp\left\{-\frac{t-t'}{\tau}-\frac{[x-X(t')]^2}{2\lambda^2}\right\}\right)\Bigg|_{x=X(t)} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\left[x-X(t')\right]\exp\left\{-\frac{t-t'}{\tau}-\frac{[x-X(t')]^2}{2\lambda^2}\right\}\Bigg|_{x=X(t)} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\left[X(t)-X(t')\right]\exp\left\{-\frac{t-t'}{\tau}-\frac{[X(t)-X(t')]^2}{2\lambda^2}\right\} \ . \end{aligned}$$

We then use the steady field-drive condition $X(t) - X(t') = W(t - t')$ at $t \to +\infty$ and define $t'' = t - t'$, so that the temporal integration $\int dt''$ will run from 0 to $+\infty$:

$$\begin{aligned} W &= \frac{\kappa\gamma}{\lambda^2}\int_0^t dt'\left[W(t-t')\right]\exp\left\{-\frac{t-t'}{\tau}-\frac{[W(t-t')]^2}{2\lambda^2}\right\} \\ &= \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt''(Wt'')\exp\left[-\frac{t''}{\tau}-\frac{(Wt'')^2}{2\lambda^2}\right] \ . \end{aligned} \tag{41}$$

For the set of parameter values $(\kappa, \tau, \gamma, \lambda)$ when the field-drive mechanism start to emerge, we can treat the field-drive velocity as infinitesimal small $W = 0^+$. Thus, divide both sides of Eq.(41) by W, we can arrive at:

$$1 = \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt' t'' \exp\left[-\frac{t''}{\tau} - \frac{(Wt'')^2}{2\lambda^2}\right]\Bigg|_{W=0^+} = \frac{\kappa\gamma}{\lambda^2}\int_0^{+\infty} dt' t'' \exp\left(-\frac{t''}{\tau}\right) = \frac{\kappa\gamma\tau^2}{\lambda^2} \quad .$$

Hence, the smallest guiding coefficient that give us field-drive, for $\tau = 50\text{s}$, $\gamma = 1\text{s}^{-1}$, $\lambda = 10\mu\text{m}$:

$$\kappa = \frac{\lambda^2}{\gamma\tau^2} = \boxed{4 \times 10^{-2}\mu\text{m/s}} \quad .$$