2022 Online Physics Olympiad: Open Contest

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before June 13, 2022**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11} \ (\mathrm{N \cdot m^2})/\mathrm{kg^2}$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30} \ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1 \ \mathrm{u} = 1.66 \cdot 10^{-27} \ \mathrm{kg} = 931 \ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34} \ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15} \ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12} \ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9 \ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7} \ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1 \ \mathrm{atm} = 1.01 \cdot 10^5 \ \mathrm{N/m^2} = 1.01 \cdot 10^5 \ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8} \ \mathrm{W/m^2/K^4}$$

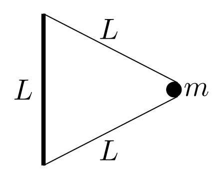
Problems

1. Two Projectiles A player throws two tennis balls on a level ground at $v = 20$ m/s in the same direction, once at an angle of $\alpha = 35°$ and once at an angle $\beta = 55°$ to the horizontal. The distance between the landing spots of the two balls is d. Find d in meters.

Assume the height of the player is negligble and ignore air resistance.

Solution 1: The range of a projectile is proportional as $R \propto \sin 2\theta$, or $R \propto \cos\theta \sin\theta$. As $\cos(90 - \theta) = \sin\theta$, and $\alpha + \beta = 90$, the distance travelled by both projectiles are the same.
$\boxed{0 \text{ m}}$

2. Bow and Arrow Consider the following simple model of a bow and arrow. An ideal elastic string has a spring constant $k = 10$ N/m and relaxed length $L = 1$ m which is attached to the ends of an inflexible fixed steel rod of the same length L as shown below. A small ball of mass $m = 2$ kg and the thread are pulled by its midpoint away from the rod until each individual part of the thread have the same length of the rod, as shown below. What is the speed of the ball in meters per seconds right after it stops accelerating? Assume the whole setup is carried out in zero gravity.



Solution 2: We can use conservation of energy. The bow string has its potential increased as

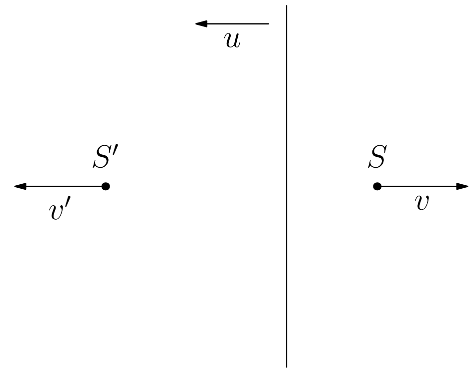
$$E_p = \frac{1}{2}k(2L - L)^2 = \frac{1}{2}kL^2.$$

This all turns into the kinetic energy of the ball $E_k = \frac{1}{2}mv^2$, so

$$E_p = E_k \implies \frac{1}{2}kL^2 = \frac{1}{2}mv^2 \implies v = L\sqrt{\frac{k}{m}}.$$

$\boxed{2.23 \text{ m/s}}$

3. City Lights A truck (denoted by S) is driving at a speed $v = 2$ m/s in the opposite direction of a car driving at a speed $u = 3$ m/s, which is equipped with a rear-view mirror. Both v and u are measured from an observer on the ground. Relative to this observer, what is the speed (in m/s) of the truck's image S' through the car's mirror? Car's mirror is a plane mirror.



Solution 3: In the mirror's frame of reference, the source speed and the image speed is both $u+v$ but in opposite direction. Now, go back the the observer's frame of reference, the image speed becomes v':

$$v' = (u+v) + u = 2u + v = 8\text{m/s} \tag{1}$$

$\boxed{8 \text{ m/s}}$

4. Springing Earth For this problem, assume the Earth moves in a perfect circle around the sun in the xy plane, with a radius of $r = 1.496 \times 10^{11}$ m, and the Earth has a mass $m = 5.972 \times 10^{24}$ kg. An alien stands far away from our solar system on the x axis such that it appears the Earth is moving along a one dimensional line, as if there was a zero-length spring connecting the Earth and the Sun.

For the alien at this location, it is impossible to tell just from the motion if it's 2D motion via gravity or 1D motion via a spring. Let U_g be the gravitational potential energy ignoring its self energy if Earth moves via gravity, taking potential energy at infinity to be 0 and U_s be the maximum spring potential energy if Earth moves in 1D via a spring. Compute U_g/U_s.

Solution 4: One naive idea is to directly compute U_g and U_s, but we can use the fact that their frequencies are the same, or:

$$\omega^2 = \frac{k}{m} = \frac{GM}{r^3} \implies kr^2 = \frac{GM}{r}$$

Then,

$$U_g = -\frac{GMm}{r} = -kr^2$$

and

$$U_s = \frac{1}{2}kr^2.$$

Therefore,

$$U_g/U_s = -2.$$

$\boxed{2}$

5. Battle Ropes Battle ropes can be used as a full body workout (see photo). It consists of a long piece of thick rope (ranging from 35 mm to 50 mm in diameter), wrapped around a stationary pole. The athlete grabs on to both ends, leans back, and moves their arms up and down in order to create waves, as shown in the photo.

The athlete wishes to upgrade from using a 35 mm diameter rope to a 50 mm diameter rope, while keeping everything else the same (rope material, rope tension, amplitude, and speed at which her arms move back and forth). By doing so, the power she needs to exert changes from P_0 to P_1. Compute P_1/P_0.

Solution 5: The power transmitted by a wave is given by

$$P = \frac{1}{2}\mu\omega^2 A^2 v,$$

where $\mu = \frac{m}{L}$ is the linear mass density, A is the amplitude, and v is the speed of the wave. The speed of a wave on a rope is given by

$$v = \sqrt{\frac{T}{\mu}}, \tag{2}$$

where T is the tension. Note that ω, A, T will all remain constant when changing the radius. Thus, $P \propto \sqrt{\mu} \propto \sqrt{m}$. As we increase the radius by a factor of $f = \frac{50}{35}$, we change the mass by f^2, so the power changes by a factor of f, giving us

$$P_1/P_0 = f = 1.43$$

$\boxed{1.43}$

6. Polarizers Given vertically polarized light, you're given the task of changing it to horizontally polarized light by passing it through a series of $N = 5$ linear polarizers. What is the maximum possible efficiency of this process? (Here, efficiency is defined as the ratio between output light intensity and input light intensity.)

Solution 6:
Let $\theta_0 = 0$ be the original direction of polarization and $\theta_5 = \pi/2$ the final direction of polarization. The 5 polarizers are directed along $\theta_1, \theta_2, \ldots, \theta_5$. Let $\delta_k = \theta_k - \theta_{k-1}$, so that the efficiency is

$$\eta = \prod_{k=1}^{5} \cos^2 \delta_k.$$

We wish to maximize η subject to the constraint that $\sum_k \delta_k = \pi/2$. Clearly, the $\delta_k's$ should be non-negative, implying that $0 \leq \delta_k \leq \pi/2$ and thus $\cos\delta_k \geq 0$ for all k.

We claim that the maximum is achieved when all δ_k are equal. If not, let $\delta_i \neq \delta_{i+1}$. Then

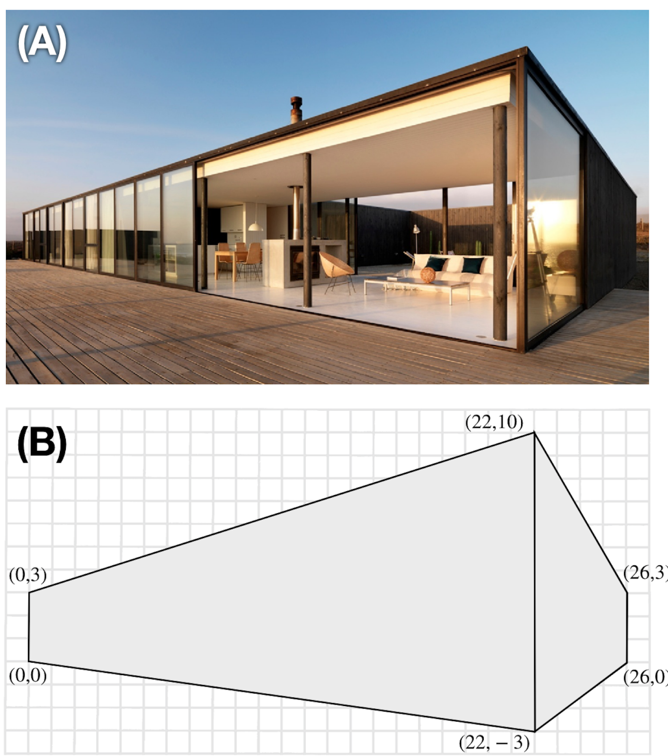
$$\begin{aligned}\cos\delta_i\cos\delta_{i+1} &= \frac{1}{2}[\cos(\delta_i+\delta_{i+1})+\cos(\delta_i-\delta_{i+1})]\\ &< \frac{1}{2}[\cos(\delta_i+\delta_{i+1})+1]\\ &= \frac{1}{2}[\cos(\delta_i'+\delta_{i+1}')+\cos(\delta_i'-\delta_{i+1}')]\end{aligned}$$

where $\delta_i' = \delta_{i+1}' = \frac{\delta_i+\delta_{i+1}}{2}$. So replacing δ_i, δ_{i+1} with δ_i', δ_{i+1}' increases η.
So η is maximized when all δ_k are equal, i.e., $\delta_k^* = \frac{\pi}{10}$ for all k. Then

$$\eta^* = \cos^{10}\left(\frac{\pi}{10}\right) \approx 0.6054.$$

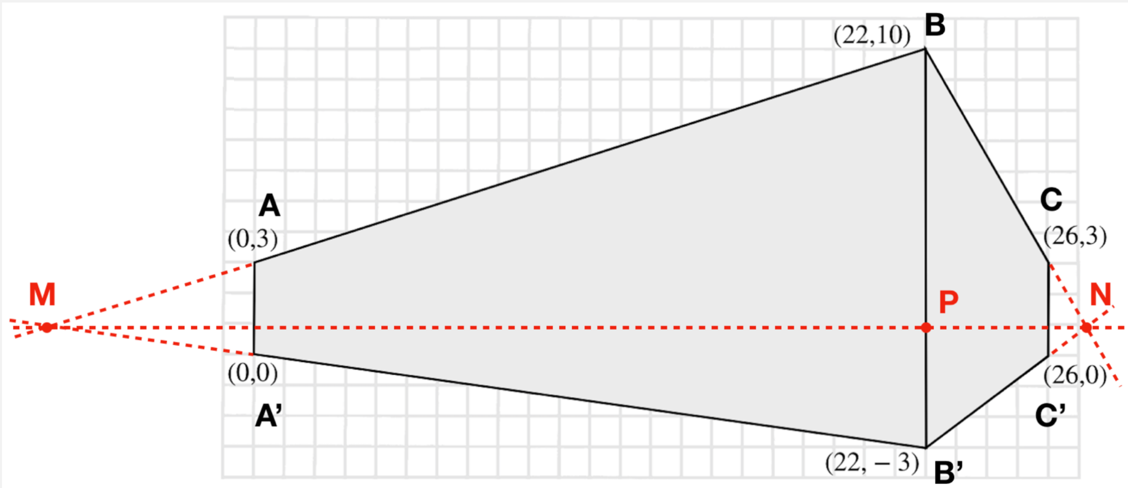
$\boxed{0.6054}$

7. Fatal Frame These days, there are so many stylish rectangular home-designs (see figure A). It is possible from the outline of those houses in their picture to estimate with good precision where the camera was. Consider an outline in one photograph of a rectangular house which has height $H = 3$ meters (see figure B for square-grid coordinates). Assume that the camera size is negligible, how high above the ground (in meters) was the camera at the moment this picture was taken?



Solution 7:
The formation of the house's image seen in the picture is due to pinhole principle, and note that the fish-eye effect here is weak (straight-lines stays straight). Define points A, B, C, A', B', C' as in the attached Fig., since AA', BB', CC' stays parallel we know that the camera looked horizontally

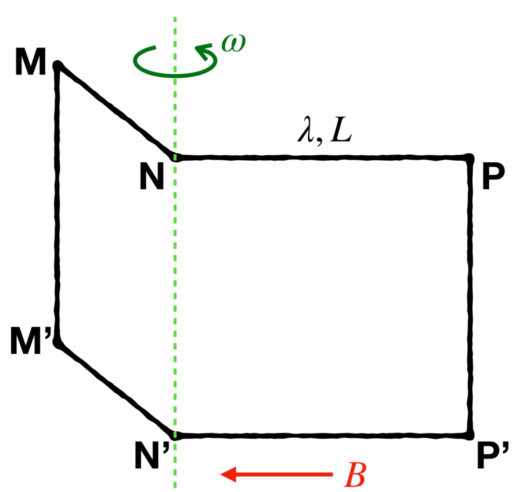
at the time this picture is taken.



To determine the height of the camera at the very same moment, we need to know the where is the horizontal plane passing through the camera in the picture which is collapsed into a line. That can be found by finding the intersection M of $AB \cap A'B'$ and the intersection N of $BC \cap B'C'$, then MN is the line of interests. MN intersects BB' at P, the position of P can be calculated too be $(22, 0.9)$, therefore the height of the camera is the length-ratio PB'/BB' times 3m, which equals to 0.9m.

$\boxed{0.9 \text{ m}}$

8. The Wire Consider a thin rigid wire-frame MNPP'N'M' in which MNN'M' and NPP'N' are two squares of side L with resistance per unit-length λ and their planes are perpendicular. The frame is rotated with a constant angular velocity ω around an axis passing through NN' and put in a region with constant magnetic field B pointing perpendicular to NN'. What is the total heat released on the frame per revolution (in Joules)? Use $L = 1$m, $\lambda = 1\Omega/$m, $\omega = 2\pi$rad/s and $B = 1$T.



Solution 8: In this setting, for every orientation during rotation the total magnetic flux passing though MNPP'N'M' is tthe same as through MPP'M', which has area $S = \sqrt{2}L^2$.

The magnetic flux is:

$$\Phi(t) = BS\sin(\omega t) = \sqrt{2}BL^2\sin(\omega t)\ . \tag{3}$$

The emf running around the wire-frame is:

$$E(t) = \frac{d}{dt}\Phi(t) = \sqrt{2}BL^2\omega\cos(\omega t)\ . \tag{4}$$

The electrical current running around the wire-frame is:

$$I(t) = \frac{E(t)}{6\lambda L} = \frac{BL\omega\cos(\omega t)}{3\sqrt{2}\lambda}\ . \tag{5}$$

The heat released power is:

$$\frac{d}{dt}Q(t) = I^2(t)\times 6\lambda L = \frac{B^2L^3\omega^2\cos^2(\omega t)}{3\lambda}\ . \tag{6}$$

Thus the total heat released per revolution is:

$$Q = \int_0^{2\pi/\omega} dt\frac{d}{dt}Q(t) = \frac{B^2L^3\omega^2\int_0^{2\pi/\omega} dt\cos^2(\omega t)}{3\lambda} = \frac{\pi B^2L^3\omega}{3\lambda} \approx 6.58\text{J}\ . \tag{7}$$

$\boxed{6.58\text{ J}}$

9. Melting Iceberg In this problem, we explore how fast an iceberg can melt, through the dominant mode of forced convection. For simplicity, consider a very thin iceberg in the form of a square with side lengths $L = 100$ m and a height of 1 m, moving in the arctic ocean at a speed of 0.2 m/s with one pair of edges parallel to the direction of motion (Other than the height, these numbers are typical of an average iceberg). The temperature of the surrounding water and air is 2°C, and the temperature of the iceberg is 0°C. The density of ice is 917 kg/m^3 and the latent heat of melting is $L_w = 334\times10^3$ J/kg.

The heat transfer rate $\dot{Q}$ between a surface and the surrounding fluid is dependent on the heat transfer coefficient h, the surface area in contact with the fluid A, and the temperature difference between the surface and the fluid ΔT, via $\dot{Q} = hA\Delta T$.

In heat transfer, three useful quantities are the Reynold's number, the Nusselt number, and the Prandtl number. Assume they are constant through and given by (assuming laminar flow):

$$\text{Re} = \frac{\rho v_\infty L}{\mu}, \qquad \text{Nu} = \frac{hL}{k}, \qquad \text{Pr} = \frac{c_p\mu}{k}$$

where:

- ρ: density of the fluid
- v_∞: speed of the fluid with respect to the object (at a very far distance)
- L: length of the object in the direction of motion

- μ: dynamic viscosity of the fluid
- k: thermal conductivity of the fluid
- c_p : the specific heat capacity of the fluid

Through experiments, the relationship between the three dimensionless numbers is, for a flat plate:

$$\mathrm{Nu} = 0.664\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}.$$

Use the following values for calculations:

	Air	Water
ρ (kg/m^3)	1.29	1000
μ (kg/(m · s))	1.729×10^{-5}	1.792×10^{-3}
c_p (J/(kg · K))	1004	4220
k (W/(m · K))	0.025	0.556

The initial rate of heat transfer is $\dot{Q}$. Assuming this rate is constant (this is not true, but will allow us to obtain an estimate), how long (in days) would it take for the ice to melt completely? Assume convection is only happening on the top and bottom faces. Round to the nearest day.

Solution 9: The heat transfer coefficient for water-ice and air-ice contact can be figured out with the relationship between the three dimensionless numbers

$$\mathrm{Nu} = 0.664\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3} \implies h = 0.664\frac{k}{L}\left(\frac{\rho v_\infty L}{\mu}\right)^{1/2}\left(\frac{c_p\mu}{k}\right)^{1/3}.$$

As

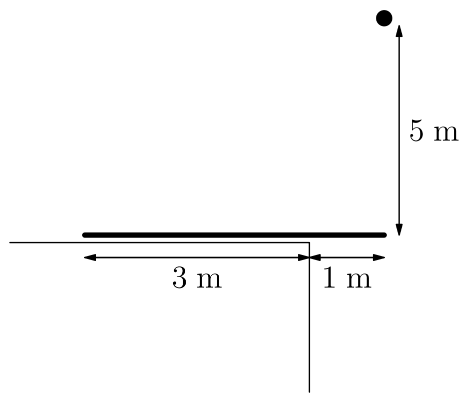
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = hA\Delta T,$$

we then have

$$t(\dot{Q}_a + \dot{Q}_w) = (h_w A_w + h_a A_a)\Delta T \implies t = \frac{\rho L^2 H L_w}{\Delta T}\frac{1}{h_w L^2 + h_a L^2} = 59.84 \approx 60 \text{ days}.$$

$\boxed{\text{60 days}}$

10. Scale A scale of uniform mass $M = 3$ kg of length $L = 4$ m is kept on a rough table (infinite friction) with $l = 1$ m hanging out of the table as shown in the figure below. A small ball of mass $m = 1$ kg is released from rest from a height of $h = 5$ m above the end of the scale. Find the maximum angle (in degrees) that the scale rotates by in the subsequent motion if ball sticks to the scale after collision. Take gravity $g = 10$ m/s^2.



Solution 10: The ball falls with velocity $\sqrt{2gh} = 10$. Applying conservation of angular momentum about the end point of the table.

$$L_i = mvx$$
$$= 10$$
$$L_f = I_1\omega$$
$$I_1 = 4 + 3 + 1 = 8$$
$$\boxed{\implies \omega = 1.25}$$

Now applying energy conservation

$$E_i = \frac{1}{2}I\omega^2 = 6.25$$
$$E_f = M_{total}gx_{com}\sin(\theta) = 20\sin\theta$$
$$\theta = 18.21^\circ$$

$\boxed{18.21^\circ}$

11. Levitating In a galaxy far, far away, there is a planet of mass $M = 6 \cdot 10^{27}$ kg which is a sphere of radius R and charge $Q = 10^3$ C uniformly distributed. Aliens on this planet have devised a device for transportation, which is an insulating rectangular plate with mass $m = 1$ kg and charge $q = 10^4$ C. This transportation device moves in a circular orbit at a distance $r = 8 \cdot 10^6$ m from the center of the planet. The aliens have designated this precise elevation for the device, and do not want the device to deviate at all. In order to maintain its orbit, the device contains a relatively small energy supply. Find the power (in Watts) that the energy supply must release in order to sustain this orbit.

The velocity of the device can be assumed to be much smaller than the speed of light, so that relativistic effects can be ignored. The device can also be assumed to be small in comparison to the size of the planet.

Solution 11: The centripetal force is given by

$$\frac{mv^2}{r} = \frac{GMm}{r^2} - \frac{qQ}{4\pi\epsilon_0 r^2},$$

which implies the centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{GM}{r^2} - \frac{qQ}{4\pi\epsilon_0 m r^2}.$$

Now, the device loses energy due to its acceleration, as given by the Larmor formula. The power needed to sustain motion is

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = 0.522 \text{ W}.$$

$\boxed{0.522 \text{ W}}$

12. Singing in the Rain A raindrop of mass $M = 0.035$ g is at height $H = 2$ km above a large lake. The raindrop then falls down (without initial velocity), mixing and coming to equilibrium with the lake. Assume that the raindrop, lake, air, and surrounding environment are at the same temperature $T = 300$ K. Determine the magnitude of entropy change associated with this process (in J/K).

Solution 12: The total heat gain is equal to the change in potential energy of the raindrop, which spreads through out the whole environment at thermally equilibrium temperature T (the environment is very large so any change in T is negligible). The entropy gain ΔS is thus generated by the dissipation of this potential energy MgH to internal energy ΔU in the environment (given that the specific volume of water doesn't change much, $\Delta U \approx MgH$). Hence the entropy change associated with this process can be estimated by:

$$S = \frac{\Delta U}{T} \approx \frac{MgH}{T} \approx 2.29 \times 10^{-3} \text{J/K} \tag{8}$$

$\boxed{2.29 \times 10^{-3} \text{ J/K}}$

13. Rocket Launch A rocket with mass of 563.17 (not including the mass of fuel) metric tons sits on the launchpad of the Kennedy Space Center (latitude 28°31′27″N, longitude 80°39′03″W), pointing directly upwards. Two solid fuel boosters, each with a mass of 68415kg and providing 3421kN of thrust are pointed directly downwards.

The rocket also has a liquid fuel engine, that can be throttled to produce different amounts of thrust and gimbaled to point in various directions. What is the minimum amount of thrust, in kN, that this engine needs to provide for the rocket to lift vertically (to accelerate directly upwards) off the launchpad?

Assume $G = 6.674 \times 10^{-11} \frac{\text{N}\cdot\text{m}^2}{\text{s}^3}$, and that the Earth is a perfect sphere of radius 6370km and mass 5.972×10^{24}kg that completes one revolution every 86164s and that the rocket is negligibly small compared to the Earth. Ignore buoyancy forces.

Solution 13: Note the additional information provided in the problem (latitude, Earth radius, revolution period), which makes it clear that the effect of the rotation of the Earth must also be considered.
We first compute local gravitational acceleration:

$$g = \frac{GM}{R^2} = 9.823 \frac{\text{m}}{\text{s}^2}$$

And also acceleration due to the Earth's rotation:

$$a = (R\cos\theta)\omega^2 = 0.02976\frac{\mathrm{m}}{\mathrm{s}^2}$$

Then vertical acceleration is:

$$g - a\cos\theta = 9.7965\frac{\mathrm{m}}{\mathrm{s}^2}$$

And horizontal acceleration:

$$a\sin\theta = 0.0142\frac{m}{s^2}$$

The total mass of the craft is 700 metric tons, so the needed forces for liftoff are:

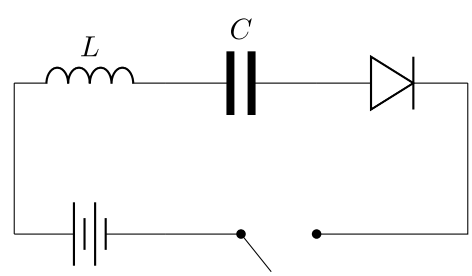
$$(F_x, F_y) = (6857527, 9948)\mathrm{N}$$

Subtracting out the effect of the solid fuel boosters, the force the liquid fuel engine needs to provide is:

$$(F_x, F_y) = (15527, 9948)\mathrm{N}$$

So the final answer is 18.44kN. $\boxed{18.44\mathrm{kN}}$

The following information applies for the next two problems. A circuit has a power source of $\mathcal{E} = 5.82\,\mathrm{V}$ connected to three elements in series: an inductor with $L = 12.5\,\mathrm{mH}$, a capacitor with $C = 48.5\,\mu\mathrm{F}$, and a diode with threshold voltage $V_0 = 0.65\,\mathrm{V}$. (Of course, the polarity of the diode is aligned with that of the power source.) You close the switch, and after some time, the voltage across the capacitor becomes constant. (*Note:* An ideal diode with threshold voltage V_0 is one whose IV characteristic is given by $I = 0$ for $V < V_0$ and $V = V_0$ for $I > 0$.)



14. LC-Diode 1 How much time (in seconds) has elapsed before the voltage across the capacitor becomes constant?

Solution 14:
When current is flowing clockwise, the circuit is equivalent to an LC circuit with a power source $\mathcal{E} - V_0$. Thus, the voltage U is sinusoidal about its equilibrium voltage $U_0 = \mathcal{E} - V_0$ with frequency $\omega = 1/\sqrt{LC}$.
When the switch is closed, $I = 0$ and $U = U_{min} = 0$. Afterwards, I increases to I_{max} and decreases back to 0, completing half a period of a sine wave. However, I can not go negative due to the presence of the diode. Instead, a reverse voltage builds up on the diode (so that the voltage across the inductor becomes 0), and I stays at 0. At this point, U becomes constant as well.
The time it took until the system became static was half a period of the LC circuit oscillation, i.e.,

$$\frac{\pi}{\omega} = \pi\sqrt{LC} = 2.446 \times 10^{-3}\ \mathrm{s}.$$

$\boxed{2.446 \times 10^{-3}\ \text{s}}$

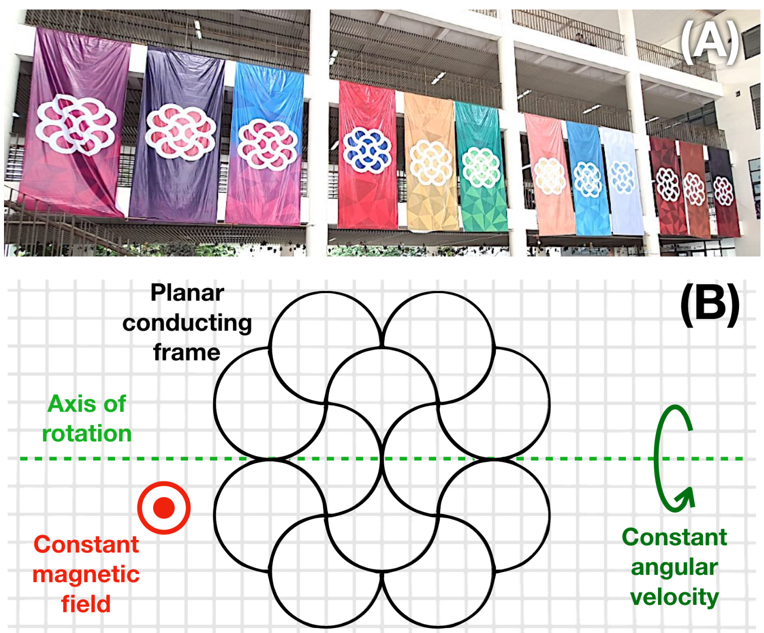
15. LC-Diode 2 What is the magnitude of final voltage (in volts) across the capacitor?

Solution 15:
As I increased from 0 to I_{max} and then decreased to 0 sinusoidally, U increased from $U_{min} = 0$ to U_{max} sinusoidally. Recall that the equilibrium $U_0 = \frac{U_{min}+U_{max}}{2} = \mathcal{E} - V_0$, so the final voltage on the capacitor is

$$U_{max} = 2U_0 = 2(\mathcal{E} - V_0) = 10.34 \text{ V}.$$

$\boxed{10.34\ \text{V}}$

16. Raging Loop At Hanoi-Amsterdam High School in Vietnam, every subject has its own flag (see Figure A, taken by Tung X. Tran). While the flags differ in color, they share the same central figure. Consider a planar conducting frame of that figure rotating at a constant angular velocity in a uniform magnetic field (see Figure B). The frame is made of thin rigid wires with same uniform curvature and same resistance per unit length. What fraction of the total heat released is released by the outermost wires?

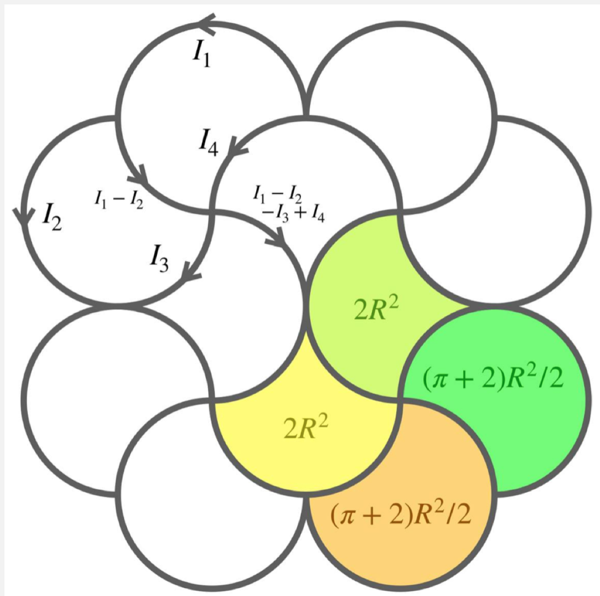


Solution 16:
We call the current looping in the wires I_1, I_2, I_3, I_4 as shown in the attached Fig., and define the resistance of every quarter-circular section (radius R) of the wires to be ρ, then considering the

EMF on every loop gives:

$$\begin{aligned}
\Big(2I_1 + (I_1 - I_2) - I_4\Big)\rho &= \frac{dB_\perp}{dt} \times \frac{\pi+2}{2}R^2 \ , \\
\Big(2I_2 - I_3 - (I_1 - I_2)\Big)\rho &= \frac{dB_\perp}{dt} \times \frac{\pi+2}{2}R^2 \ , \\
\Big(2I_4 + 2(I_1 - I_2 - I_3 + I_4)\Big)\rho &= \frac{dB_\perp}{dt} \times 2R^2 \ , \\
\Big(2I_3 - 2(I_1 - I_2 - I_3 + I_4)\Big)\rho &= \frac{dB_\perp}{dt} \times 2R^2 \ ,
\end{aligned} \tag{9}$$

in which $B_\perp$ is the perpendicular component of the magnetic field.



The set of equations in Eq. (9) can be solved to get the relation between currents:

$$I_1 = I_2 \ , \ I_3 = I_4 \ , \ \frac{I_1}{I_3} = \frac{\pi+4}{4} \ . \tag{10}$$

The fraction of heat released on the outermost wires can be calculated:

$$\frac{Q_{outermost}}{Q_{all}} = \frac{\left(8I_1^2 + 8I_2^2\right)\rho}{\left(8I_1^2 + 8I_2^2 + 4I_3^2 + 4I_4^2\right)\rho} = \frac{(\pi+4)^2}{(\pi+4)^2+8} \approx 0.864 \text{ J} \tag{11}$$

$\boxed{0.864 \text{ J}}$

17. Moon Landing A spacecraft is orbiting in a very low circular orbit at a velocity v_0 over the equator of a perfectly spherical moon with uniform density. Relative to a stationary frame, the spacecraft completes a revolution of the moon every 90 minutes, while the moon revolves in the same direction once every 24 hours. The pilot of the spacecraft would like to land on the moon using the following process:

1. Start by firing the engine directly against the direction of motion.
2. Orient the engine over time such that the vertical velocity of the craft remains 0, while the horizontal speed continues to decrease.

3. Once the velocity of the craft relative to the ground is also 0, turn off the engine.

Assume that the engine of the craft can be oriented instantly in any direction, and the craft has a TWR (thrust-to-weight ratio, where weight refers to the weight at the moon's surface) of 2, which remains constant throughout the burn. If the craft starts at $v_0 = 500$ m/s, compute the delta-v expended to land, minus the initial velocity, i.e. $\Delta v - v_0$.

Solution 17:
The trick in this question is to work in dimensionless units. Let v be the ratio of the craft's velocity to orbital velocity. Then, if the craft has horizontal velocity v, the acceleration downwards is the following:

$$a_V = \frac{v_0^2}{r} - \frac{(v_0 v)^2}{r} = \frac{v_0^2}{r}(1 - v^2) = g_m(1 - v^2)$$

As the TWR is 2, the total acceleration the engine provides is $a_0 = 2g_m$, where g_m is the surface gravity of the moon. As this total acceleration is the sum of horizontal and vertical components, and the vertical component cancels out the downwards acceleration:

$$a_H = \sqrt{(2g_m)^2 - (g_m(1 - v^2))^2} = g_m\sqrt{2^2 - (1 - v^2)^2}$$

And a_H is related to v (which is dimensionless!) by the following relation:

$$\frac{\mathrm{d}(v_0 v)}{\mathrm{d}t} = -a_H$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{g_m}{v_0}\sqrt{2^2 - (1 - v^2)^2}$$

$$\frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = -\frac{g_m}{v_0}\mathrm{d}t$$

At the start, $v = 1$. However, at landing, velocity of the craft is 0 relative to the surface, not a stationary frame! Therefore, we use the orbital periods data to determine the final v to be $\frac{1.5}{24} = \frac{1}{16}$. Then integrating:

$$\int_{v=1}^{v=1/16} \frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = -\frac{g_m}{v_0}t$$

And as delta-v is related to time by $\Delta v = a_0 t$:

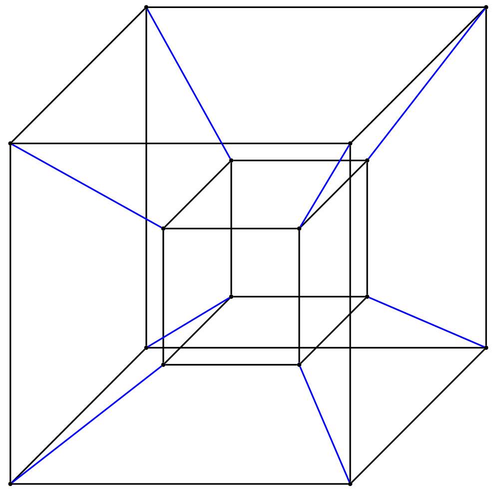
$$\Delta v = a_0 t = a_0 \frac{v_0}{g_m} \int_{1/16}^{1} \frac{\mathrm{d}v}{\sqrt{2^2 - (1 - v^2)^2}} = a_0 \frac{v_0}{a_0} 2(0.503) = 500 \cdot 1.006 = 503.06\frac{\mathrm{m}}{\mathrm{s}}.$$

And subtracting:

$$\Delta v - v_0 = 3.06\frac{\mathrm{m}}{\mathrm{s}}$$

$\boxed{3.06 \text{ m/s}}$

18. Tesseract Oscillations A tesseract is a 4 dimensional example of cube. It can be drawn in 3 dimensions by drawing two cubes and connecting their vertices together as shown in the picture below:

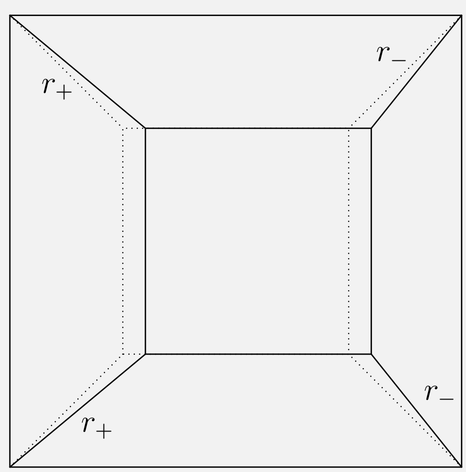


Now for the 3D equivalent. The lines connecting the vertices are replaced with ideal springs of constant $k = 10$ N/m (in blue in the figure). Now, suppose the setup is placed in zero-gravity and the outer cube is fixed in place with a sidelength of $b = 2$ m. The geometric center of the inner cube is placed in the geometric center of the outer cube, and the inner cube has a side-length $a = 1$ m and mass $m = 1.5$ kg. The inner cube is slightly displaced from equilibrium. Consider the period of oscillations

- T_1: when the springs have a relaxed length of 0;
- T_2: when the springs are initially relaxed before the inner cube is displaced.

What is $T_1 + T_2$?

Solution 18: First let us prove that there is a net external torque of $\vec{\tau} = 0$ on the cube for small displacements which means the inner cube behaves like a point mass. Consider a simple case when the cube is pushed to one side.



If we label the vertices of the cube from 1 to 4 clockwise, where 1 is the top left side, it is apparent that sides 1 and 2 provide a positive torque while sides 3 and 4 provide a negative torque. As the displacement is small, the angles created are small enough such that $\sin\theta \approx \theta$. As force is proportional to the extension of the spring as $F \propto x$, we can write that

$$\tau \propto \theta(r_+ + r_- - r_+ - r_-) \propto 0.$$

If torque is zero when the cube is displaced in the x-direction, then by symmetry, the torque is zero when the cube is displaced in the y-direction. Superposing both solutions implies that torque as a function of displacements in the x and y directions $\alpha\hat{x} + \beta\hat{y}$ is

$$\tau(\alpha x + \beta y) = \tau(\alpha x) + \tau(\beta y) = \alpha\tau(x) + \beta\tau(y) = 0.$$

1. Label the vertices of the outer cube as $1, 2, \ldots, 8$ and the vectors that point to these vertices from the inner cube as $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_2$. Consider when the inner cube deviates from equilibrium with a vector $\vec{r}$. The force as a function of $\vec{r}$ is

$$\begin{aligned} F(\vec{r}) &= k[(\vec{r}_1 - \vec{r}) + (\vec{r}_2 - \vec{r}) + \ldots (\vec{r}_8 - \vec{r})]. \\ &= k\left(\sum_{i=1}^{8} \vec{r}_i - 8\vec{r}\right) \\ &= -8k\vec{r} \end{aligned}$$

   This implies the period of oscillations is

$$T_1 = 2\pi\sqrt{\frac{m}{8k}}.$$

2. Let the center of the inner cube be $(0, 0, 0)$. Consider the coordinates $(a/2, a/2, a/2)$ and $(b/2, b/2, b/2)$ which correspond to the vertex of the inner and larger cube respectively. Consider moving the cube in the x-direction. From defining $y = b/2 - a/2$, the compressional/extension of each spring $\pm\Delta\ell$ is then

$$\begin{aligned} \Delta\ell &= \pm\sqrt{(x+y)^2 + 2y^2} - \sqrt{3}y \\ &= \pm\sqrt{3}y\sqrt{1 + \frac{2x}{3y} + \mathcal{O}(x^2)} - \sqrt{3}y \\ &\approx \pm\sqrt{3}y\frac{x}{3y} \\ &= \pm\frac{x}{\sqrt{3}}. \end{aligned}$$

   The total energy in all springs together are then

$$E = 8 \times \frac{1}{2}k\left(\frac{x}{\sqrt{3}}\right)^2 \implies F = -\frac{8k}{3}x \implies T_2 = 2\pi\sqrt{\frac{3m}{8k}}.$$

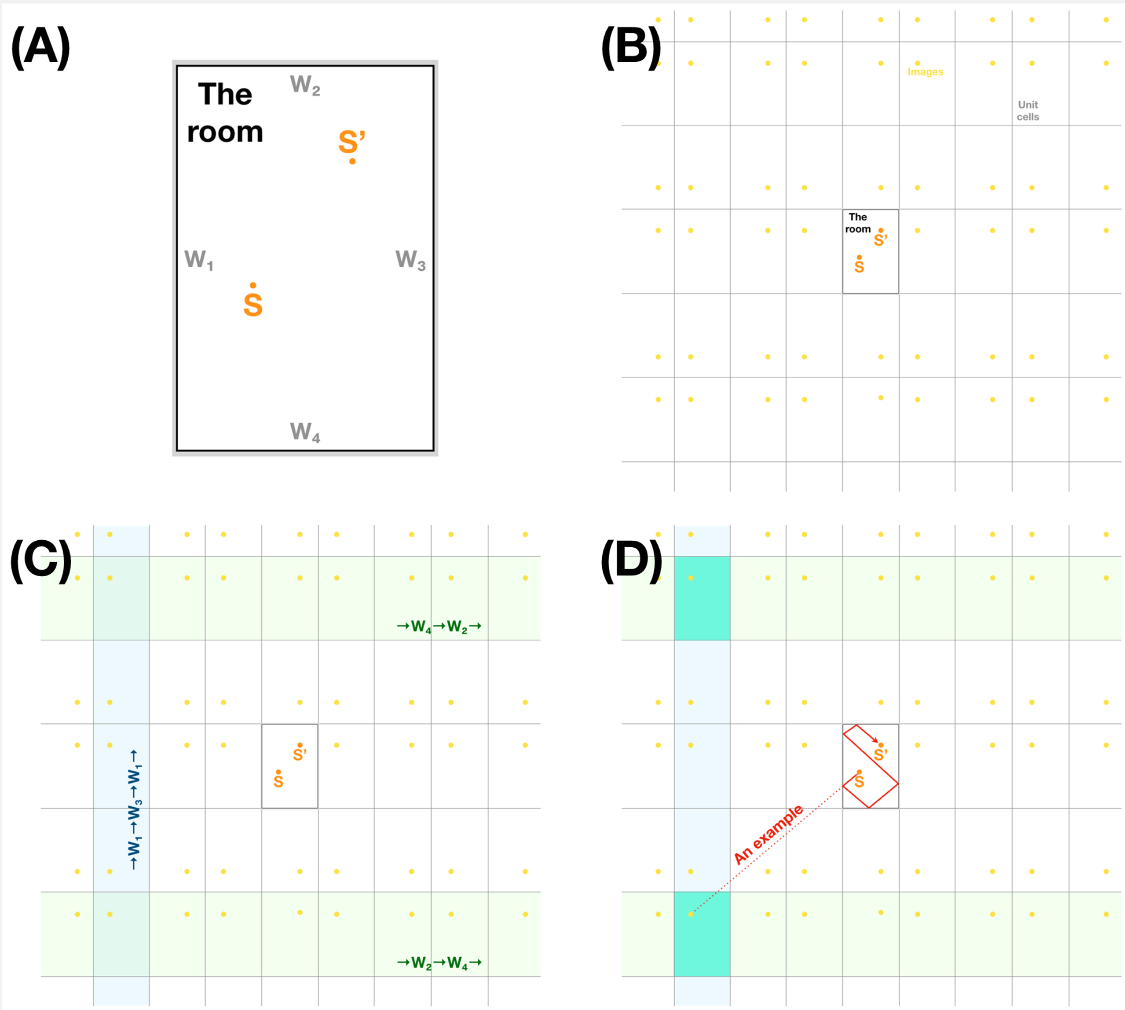
   Hence, our total answer is

$$T_1 + T_2 = 2\pi(1 + \sqrt{3})\sqrt{\frac{m}{8k}}.$$

$\boxed{2.35 \text{ s}}$

19. The Room Consider two points S and S' randomly placed inside a D-dimensional hyper-rectangular room with walls that are perfect-reflecting $(D-1)$-dimensional hyper-plane mirrors. How many different light-rays that start from S, reflect N times on one of the walls and $N-1$ times on each of the rest, then go to S'? Use $D = 7$ and $N = 3$.

Solution 19:
Using the hyper-rectangular room as the fundamental unit-cell of an infinite hyper-grid in space, then find all possible positions of S'-images through reflections: we can realize that there is only one position of S'-image inside every unit-cell.



Consider two opposite hyper-plane mirrors, since the rest of the mirrors are perpendicular to them, the numbers of reflections on them for any light-path traveled from point S and point S' can only different by 1 or less. If the numbers are both equal and non-zero, then the available positions S'-image the light-path from S should reach are two unit-cell hyper-rows that parallel to the mirrors. If the numbers are not equal, then the available positions S'-image the light-path from S should reach are one unit-cell hyper-rows that that parallel to the mirrors.

Say, without loss of generality, pick one mirror to be reflected N times and the rest to be reflected $N-1$ times each, then the number of light-rays for that pick should equal to the number of unit-cell intersections between all relevant unit-cell hyper-rows, which is half the total number of vertices a hyper-rectangular has, thus 2^{D-1}. There are $2D$ walls, thus the total number of light-rays that satisfies the task given is $2D \times 2^{D-1} = D2^D$, independent of N for all values $N > 1$.

To illustrate the above explanation, let's take a look at the simple case of $D = 2$ and $N = 2 > 1$. Consider a rectangular room, with random points S, S' and $2D = 4$ walls W_1, W_2, W_3, W_4 (see Fig. A). Without loss of generality, we want to find light-rays that go from S, reflect $N = 2$ times on W_1 and $N - 1 = 1$ times on W_2, W_3, W_4 then come to S'. Each image of S' is an unique point in a unit-cell generated by the room (see Fig. B). Note that every light-ray from S that reach S'-images in the $\to W_2 \to W_4 \to$ and $\to W_4 \to W_2 \to$ unit-cell green-rows will satisfy the requirement of one reflection on each of W_2 and W_4, every light-ray from S that reach S'-images in the $\to W_1 \to W_3 \to W_1 \to$ unit-cell blue-rows will satisfy the requirement of two reflection on W_1 and one reflection on W_3 (see Fig. C). The intersection of these rows are two unit-cells, corresponds two possible images thuss two possible light-rays that satisfies the requirement (see Fig. D).

For $D = 7$ and $N = 3 > 1$, we get 896 light-rays.

$\boxed{\text{895 light-rays}}$

* This puzzle was created with helps from Long T. Nguyen.

20. Two Rings Two concentric isolated rings of radius $a = 1$ m and $b = 2$ m of mass $m_a = 1$ kg and $m_b = 2$ kg are kept in a gravity free region. A soap film of surface tension $\sigma = 0.05\text{Nm}^{-1}$ with negligible mass is spread over the rings such that it occupies the region between the rings. The smaller ring is pulled slightly along the axis of the rings. Find the time period of small oscillation in seconds.

Solution 20: The force on the two rings when they are a distance L apart follows as

$$F = 4\pi r\sigma \sin\theta$$

In small displacements, the change in θ is small. Therefore,

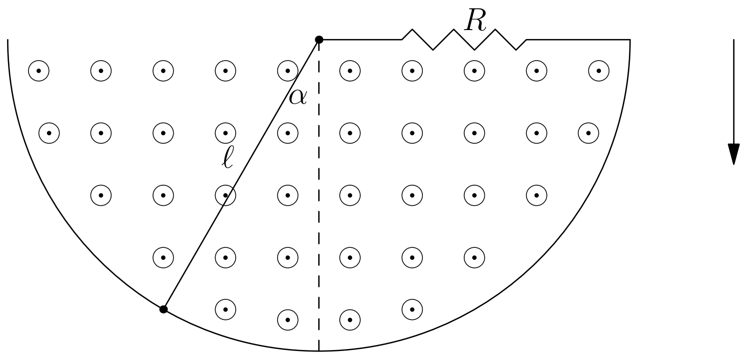
$$F = 4\pi r\sigma\theta$$
$$\frac{F}{4\pi r\sigma} = \frac{\mathrm{d}y}{\mathrm{d}r}$$
$$\int_a^b \frac{F}{4\pi\sigma r}\mathrm{d}r = \int_0^L \mathrm{d}y$$
$$\frac{F}{4\pi\sigma}\ln(b/a) = L$$

Now let a_1 and a_2 be acceleration of a and b respectively. We have that

$$a_{net} = a_1 + a_2 \tag{12}$$
$$a_{net} = F\left(\frac{m_1 + m_2}{m_1 m_2}\right) \tag{13}$$
$$\cancel{L}\omega^2 = \frac{4\pi\sigma}{\ln(b/a)}\cancel{L}\left(\frac{m_1 + m_2}{m_1 m_2}\right) \tag{14}$$
$$T = 2\pi\sqrt{\frac{\ln(b/a)m_1 m_2}{4\pi\sigma(m_1 + m_2)}} \tag{15}$$
$$T = 2\pi\sqrt{\frac{10\ln(2)}{3\pi}} = 5.388 \text{ s} \tag{16}$$

$\boxed{5.388 \text{ s}}$

21. Pendulum Circuit An open electrical circuit contains a wire loop in the shape of a semi-circle, that contains a resistor of resistance $R = 0.2\Omega$. The circuit is completed by a conducting pendulum in the form of a uniform rod with length $\ell = 0.1$ m and mass $m = 0.05$ kg, has no resistance, and stays in contact with the other wires at all times. All electrical components are oriented in the yz plane, and gravity acts in the z direction. A constant magnetic field of strength $B = 2$ T is applied in the $+x$ direction.



Ignoring self inductance and assuming that $\alpha \ll 1$, the general equation of motion is in the form of $\theta(t) = A(t)\cos(\omega t + \varphi)$, where $A(t) \geq 0$. Find ω^2.

Solution 21: The area enclosed by the wire loop is

$$A = \frac{1}{2}\ell^2\alpha + A_0$$

for small angles α, and A_0 is a constant number (which gets ignored since we really care about how this angle is changing). The flux is $\Phi = BA$ and from Lenz's Law, we have,

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{1}{2}B\ell^2\dot{\alpha}.$$

One can verify that if α is increasing, the current will flow in the clockwise direction, so we set the counterclockwise direction as positive. The current through the wire is thus,

$$i = \frac{\varepsilon}{R} = -\frac{B\ell^2}{2R}\dot{\alpha}.$$

The magnetic force acting on it is $F_B = iB\ell$ and the resulting torque is

$$\tau_B = F_B\frac{\ell}{2} = -\frac{B^2\ell^4}{4R}\dot{\alpha}.$$

Please verify that the sign is correct. The gravitational torque is $\tau_g = -mg\frac{\ell}{2}\alpha$, so the torque equation gives us

$$0 = \frac{1}{3}m\ell^2\ddot{\alpha} + \frac{B^2\ell^4}{4R}\dot{\alpha} + mg\frac{\ell}{2}\alpha$$

$$0 = \ddot{\alpha} + \frac{3}{4}\frac{B^2\ell^2}{mR}\dot{\alpha} + \frac{3}{2}\frac{g}{\ell}\alpha$$

Recall that for a damped harmonic oscillator in the form of $\ddot{\alpha} + \gamma\dot{\alpha} + \omega_0^2\alpha = 0$, the frequency of oscillations is $\omega^2 = \omega_0^2 - \gamma^2/4$, so in our case, we have

$$\omega^2 = \frac{3}{2}\frac{g}{\ell} - \frac{9}{64}\left(\frac{B^2\ell^2}{mR}\dot{\alpha}\right)^2 = 145 \text{ s}^{-1}$$

$\boxed{145 \text{ s}^{-1}}$

22. Broken Table A table of unknown material has a mass $M = 100$ kg, width $w = 4$ m, length $\ell = 3$ m, and 4 legs of length $L = 0.5$ m with a Young's modulus of $Y = 1.02$ MPa at each of the corners. The cross-sectional area of a table leg is approximately $A = 1$ cm^2. The surface of the table has a coefficient of friction of $\mu = 0.1$. A point body with the same mass as the table is put at some position from the geometric center of the table. What is the minimum distance the body must be placed from the center such that it slips on the table surface immediately after? Report your answer in centimeters.

The table surface and floor are non-deformable.

Solution 22: This problem requires some 3 dimensional reasoning. Suppose $\mathbf{s} = (s_x, s_y)$ is the gradient of the table. We can use this to calculate the additional torque from the displacement of the mass. The forces from each table leg are

$$F_i = \frac{YA}{L}\left(\pm s_x \frac{\ell}{2} \pm s_y \frac{w}{2}\right).$$

Taking the cross product as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F_i}$ shows that torque is given as

$$\boldsymbol{\tau} = \frac{YA}{L}\begin{pmatrix} -s_y w^2 \\ s_x \ell^2 \end{pmatrix}$$

which must balance out the torque Mgd from a point mass. Hence, rewriting yields

$$d = \frac{YA}{MgL}\sqrt{s_y^2 w^4 + s_x^2 \ell^4}.$$

Furthermore, note that the angle required from slipping is given from a force analysis as

$$mg\sin\theta = \mu mg\cos\theta \implies \mu = \tan\theta = |\mathbf{s}| = \sqrt{s_x^2 + s_y^2}.$$

When $w > \ell$, we can rewrite

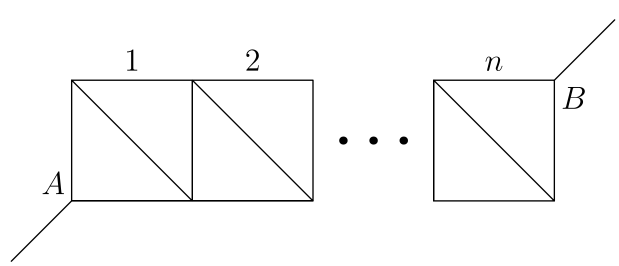
$$s_y^2 w^4 + s_x^2 \ell^4 = s_y^2(w^4 - \ell^4) + \ell^4 \mu^2$$

which is minimized to $\ell^4\mu^2$ when $s_y = 0$. Hence, we obtain

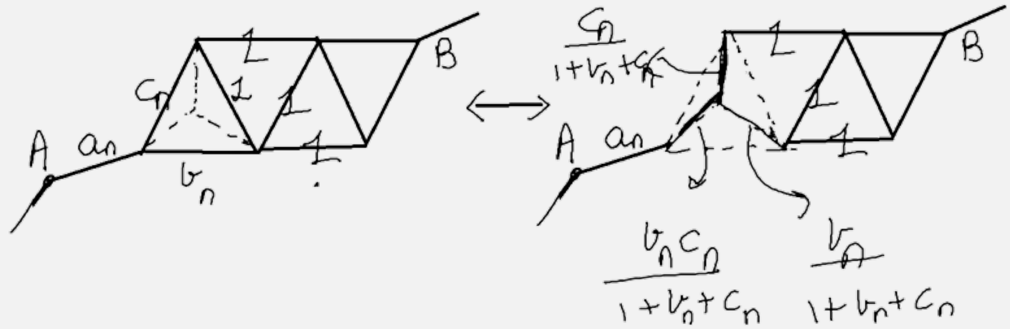
$$d = \frac{\mu \ell^2 YA}{MgL}.$$

$\boxed{18.71 \text{ m}}$

23. Resistance box In the figure below, the resistance of each wire (side and diagonal) is 1Ω. Find the value of $p + q$ if $\lim_{n\to\infty} \frac{R_{AB}}{n} = \frac{p}{q}$ where p and q are co-prime integers.



Solution 23: The circuit in the question can be redrawn after applying 2n times delta star



$$\Longrightarrow a_{n+1} = a_n + \frac{b_n c_n}{1+b_n+c_n} \tag{17}$$
$$b_{n+1} = 1 + \frac{c_n}{1+b_n+c_n}$$
$$c_{n+1} = \frac{b_n}{1+b_n+c_n}$$

For $n \to \infty$, $b_n = b_{n+1}$ and $c_n = c_{n+1}$

$$\Longrightarrow b = 1 + \frac{c}{1+b+c} \tag{18}$$
$$c = \frac{b}{1+b+c} \tag{19}$$

On solving above equations we get $b = \frac{1}{2}\left(1 + \frac{3}{\sqrt{5}}\right)$ and $c = \frac{1}{\sqrt{5}}$. Therefore we get $a_{n+1} = a_n + \frac{2}{5}$ hence $\lim_{n\to\infty} \frac{R_{AB}}{n} = \frac{2}{5}$ $\boxed{\frac{2}{5}}$

24. Dipole Conductor An (ideal) electric dipole of magnitude $p = 1 \times 10^{-6}$ C·m is placed at a distance $a = 0.05$ m away from the center of an uncharged, isolated spherical conductor of radius $R = 0.02$ m. Suppose the angle formed by the dipole vector and the radial vector (the vector pointing from the sphere's center to the dipole's position) is $\theta = 20^o$. Find the (electrostatic) interaction energy between the dipole and the charge induced on the spherical conductor.

Solution 24:
We can use the fact that if a charge Q is placed at a distance a from a grounded, conducting sphere of radius R, as far as the field outside the sphere is concerned, there is an image charge of magnitude $-Q\frac{R}{a}$ at a position $\frac{R^2}{a}$ from the origin, on the line segment connecting the origin and charge Q. It is straightforward to check that indeed, the potential on the sphere due to this image charge prescription is 0. If the point charge is instead a dipole p, we can think of this as a superposition of 2 point charges, and use the fact above. In particular, there is one charge $-Q$ at point $(a, 0, 0)$ and another charge Q at point $(a + s\cos\theta, s\sin\theta, 0)$, where s is small and $Qs = p$. Note that the dipole points in the direction θ above the x-axis. Consequently, there will be an image charge at $(\frac{R^2}{a}, 0, 0)$ with magnitude $Q\frac{R}{a}$ and an image charge at $(\frac{R^2}{a+s\cos\theta}, \frac{R^2 s\sin\theta}{a(a+s\cos\theta)}, 0)$ with magnitude $-Q\frac{R}{a+s\cos\theta}$. The image charges are close to each other but do not cancel out exactly,

so they can be represented as a superposition of an image point charge Q' and an image dipole p'. The image point charge has magnitude $Q' = -QR(\frac{1}{a+s\cos\theta} - \frac{1}{a}) = \frac{QRs\cos\theta}{a^2}$. The image dipole has magnitude $p' = Q\frac{R}{a} * \frac{R^2 s}{a^2} = \frac{QR^3 s}{a^3}$ and points towards the direction θ below the positive x-axis. Finally, since the sphere in the problem is uncharged instead of grounded, to ensure the net charge in the sphere is 0, we place another image charge $-Q'$ at the origin.
Now we can calculate the desired interaction energy, which is simply the interaction energy between the image charges and the real dipole. Using the dipole-dipole interaction formula, the interaction between the image dipole and the real dipole is given by:

$$U_1 = \frac{kpp'}{(a - \frac{R^2}{a})^3}(\cos(2\theta) - 3\cos^2\theta)$$

The interaction between the image charge at the image dipole's position and the real dipole is given by:

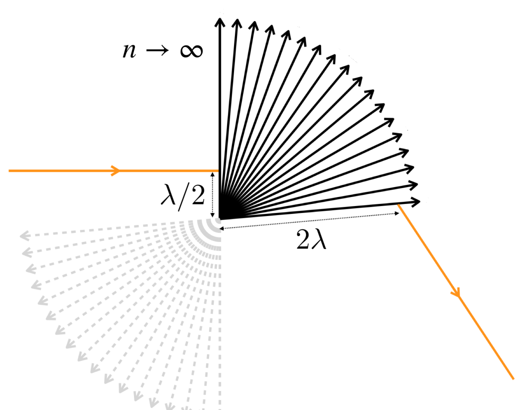
$$U_2 = -\frac{kpQ'\cos\theta}{(a - \frac{R^2}{a})^2}$$

The interaction between the image charge at the center and the real dipole is given by:

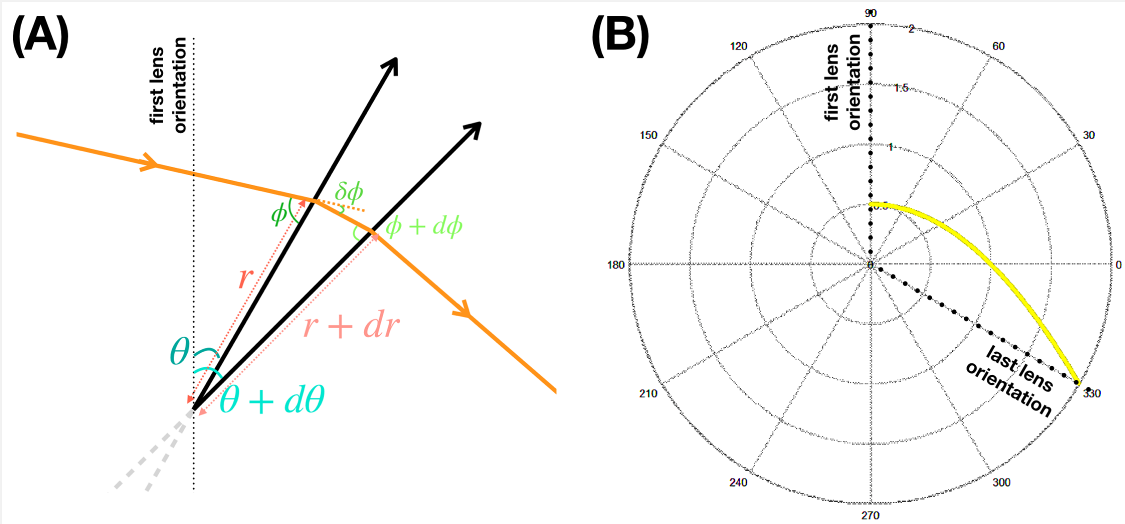
$$U_3 = \frac{kpQ'\cos\theta}{a^2}$$

The final answer is $U = U_1 + U_2 + U_3 = \frac{p^2 R}{4\pi\epsilon_0}(\frac{cos^2\theta}{a^4} - \frac{a^2\cos^2\theta + R^2}{(a^2 - R^2)^3}) = -25.22$ J. $\boxed{-25.22 \text{ J}}$

25. Dying Light Consider an optical system made of many identical ideal (negligible-thickness) half-lenses with focal length $f > 0$, organized so that they share the same center and are angular-separated equally at density n (number of lenses per unit-radian). Define the length-scale $\lambda = f/n$. A light-ray arrives perpendicular to the first lens at distance $\lambda/2$ away from the center, then leaves from the last lens at distance 2λ away from the center. Estimate the total deflection angle (in rad) of the light-ray by this system in the limit $n \to \infty$.



Solution 25: We define the angles as in Fig. A. The light-path inside the optical system is $r(\theta)$, and the angle between the first and last lens is Θ (which is an unknown but can be uniquely determined from know information).



Consider two consecutive lens at angle θ and $\theta + d\theta$, in which $d\theta = 1/n \to 0$ in the continuum limit $n \to \infty$. From the ideal-lens' equation, using the approximation that f is very large compare to other relevant length-scales in this optical setting:

$$\frac{1}{f} = \frac{1}{r\tan\phi} + \frac{1}{r\tan(\pi - \phi - \delta\phi)} \approx \frac{\delta\phi}{r\sin^2\phi} \;\Rightarrow\; \delta\phi \approx \frac{r}{f}\sin^2\phi \;, \tag{20}$$

the differential equation for the angle of arrival ϕ can be written as:

$$d\phi = \delta\phi - d\theta \;\Rightarrow\; \frac{d\phi}{d\theta} = \frac{r}{f/n}\sin^2\phi - 1 = \frac{r}{\lambda}\sin^2\phi - 1 \;. \tag{21}$$

We also have the differential relation between radial position $r(\theta)$ of the light-path and the angle of arrival ϕ as followed:

$$\frac{dr}{d\theta} = r\cot\phi \;. \tag{22}$$

From Eq. (21) and Eq. (22), we arrive at:

$$\frac{d\phi}{dr} = \frac{\frac{r}{\lambda} - 1}{r\cot\phi} \;. \tag{23}$$

Define $\zeta = \tan\phi$, then Eq. (23) becomes:

$$\frac{d\phi}{dr} = \frac{1}{1+\zeta^2}\frac{d\zeta}{dr} = \frac{\frac{r}{\lambda}\frac{\zeta^2}{1+\zeta^2} - 1}{r/\zeta} \;\Rightarrow\; -\frac{d\zeta}{\zeta^3 dr} - \frac{1}{\zeta^2 r} = \frac{1}{r} - \frac{1}{\lambda} \;. \tag{24}$$

Define $\xi = 1/\zeta^2 = 1/\tan^2\phi$, then Eq. (23) gives:

$$\frac{d\zeta}{\zeta^2 dr} = -\frac{1}{2}\frac{d\xi}{dr} \;\Rightarrow\; \frac{d\xi}{dr} - \frac{2}{r}\xi = 2\left(\frac{1}{r} - \frac{1}{\lambda}\right) \;\Rightarrow\; \frac{d}{dr}\left(\frac{\xi}{r^2}\right) = \frac{2}{r^2}\left(\frac{1}{r} - \frac{1}{\lambda}\right) \;. \tag{25}$$

Integrating both sides, then up to a constant value C, Eq. (25) gives:

$$\frac{\xi}{r^2} = -\frac{1}{r^2} + \frac{2}{\lambda r} + C \;\Rightarrow\; \xi = -1 + 2\frac{r}{\lambda} + C\frac{r^2}{\lambda^2} \;. \tag{26}$$

At $\theta = 0$, $r = \lambda/2$ and $\phi = \pi/2$ (thus $\xi = 0$), we can determine $C = 0$. Hence:

$$\cot\phi = \sqrt{2\frac{r}{\lambda} - 1} \; . \tag{27}$$

Plug Eq. (27) into Eq. (22):

$$\frac{dr}{d\theta} = \frac{r}{\lambda}\sqrt{2\frac{r}{\lambda} - 1} \;\Rightarrow\; \theta = 2\arctan\sqrt{2\frac{r}{\lambda} - 1} \; . \tag{28}$$

At $\theta = \Theta$, $r = 2\lambda$ therefore we can use Eq. (28) to get:

$$\Theta = 2\arctan\sqrt{3} = \frac{2\pi}{3} \; . \tag{29}$$

Using Eq. (27), the deflection angle Δ can be calculated to be:

$$\Delta = \Theta - \phi\Big|_{r=\lambda/2} + \phi\Big|_{r=2\lambda} = \Theta - \frac{\pi}{2} + \operatorname{arccot}\sqrt{3} = \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{\pi}{3} \approx 1.05\text{rad} \; . \tag{30}$$

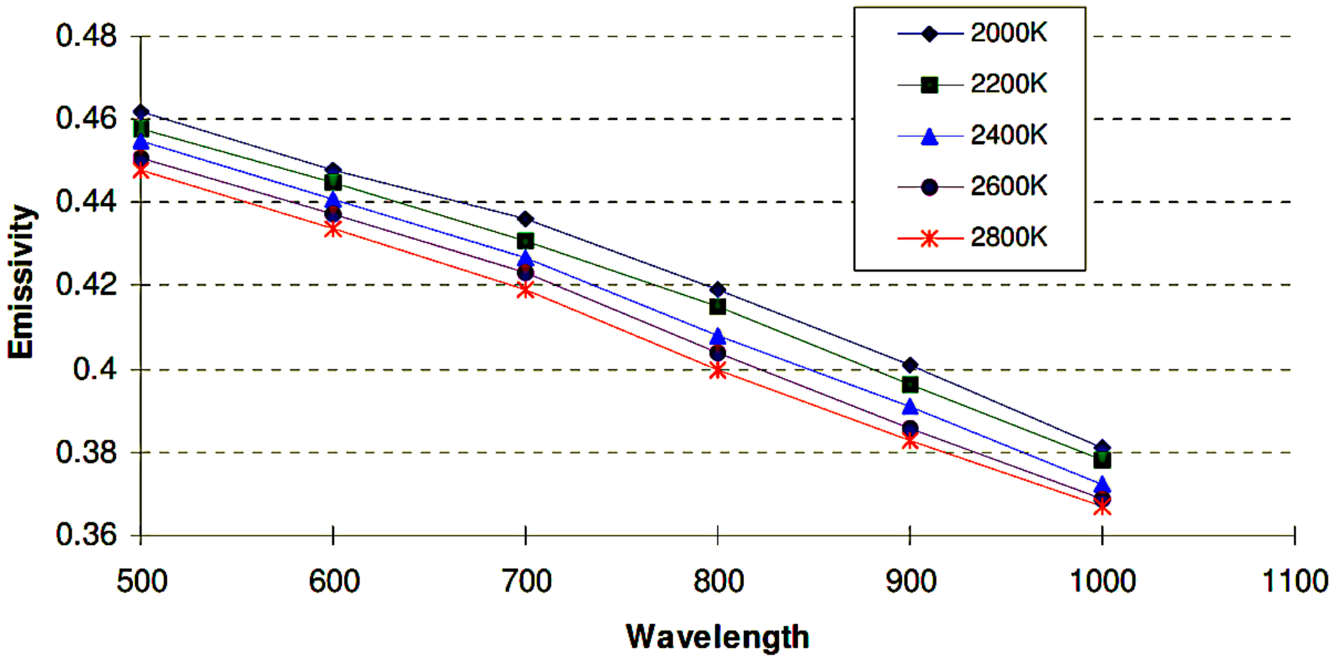
For the sake of completeness, we provide the simulated light-path inside the optical system where $n = 1000$ using MatLab (which is in great agreement with our theoretical analysis).
$\boxed{1.05 \text{ rad}}$

* This puzzle was created with helps from Long T. Nguyen.

26. Tungsten For black body radiation, Wien's Displacement Law states that its spectral radiance will peak at

$$\lambda_{\text{peak}} = \frac{b}{T}.$$

where $b = 2.89777 \times 10^{-3}$ mK, and T is the temperature of the object. When QiLin tried to reproduce this in a lab, by working with a tungsten-filament lightbulb at 2800 K, he computed a different value for b by measuring the peak wavelength using a spectrometer and multiplying it with the temperature. He hypothesizes that this discrepancy is because tungsten is not an ideal black body. The graph below, courtesy of the CRC Handbook of Chemistry and Physics, shows the emissivity of tungsten at various conditions (the units for wavelength is nm).



Assuming QiLin's hypothesis is correct, and assuming there were no other errors in the experiment, how off was his value for b? Submit $\frac{|b_{\text{theory}}-b_{\text{experiment}}|}{b_{\text{theory}}}$ as a decimal number, to *one* significant digit (giving you room to estimate where the points are).

Solution 26: Recall Planck's Law, which says the spectral radiance of a black body is given by

$$B_0(\lambda, T) = \frac{2hc^3}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

The regular Wien's Displacement Law can be derived by finding the peak by computing $\frac{\partial B_0}{\partial \lambda}$, to find the wavelength associated with the maximal radiance. For a nonideal body with emissivity $\epsilon(\lambda, T)$, we can write the radiance as

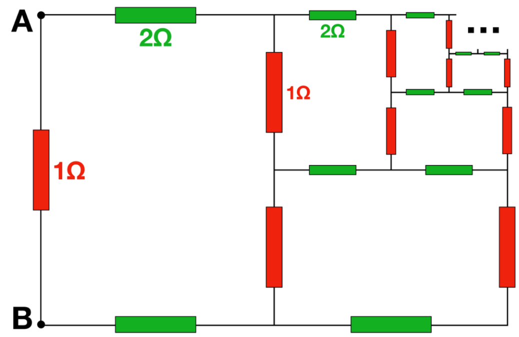
$$B(\lambda, T) = B_0(\lambda, T)\epsilon(\lambda, T).$$

We can estimate $\epsilon(\lambda, T)$ by looking at the given graph. The tungsten is at 2800K, so we will use the red line, and assuming it is near a black body, the peak wavelength should be around 1000 nm. Performing a linear approximation around 1000 nm, we get

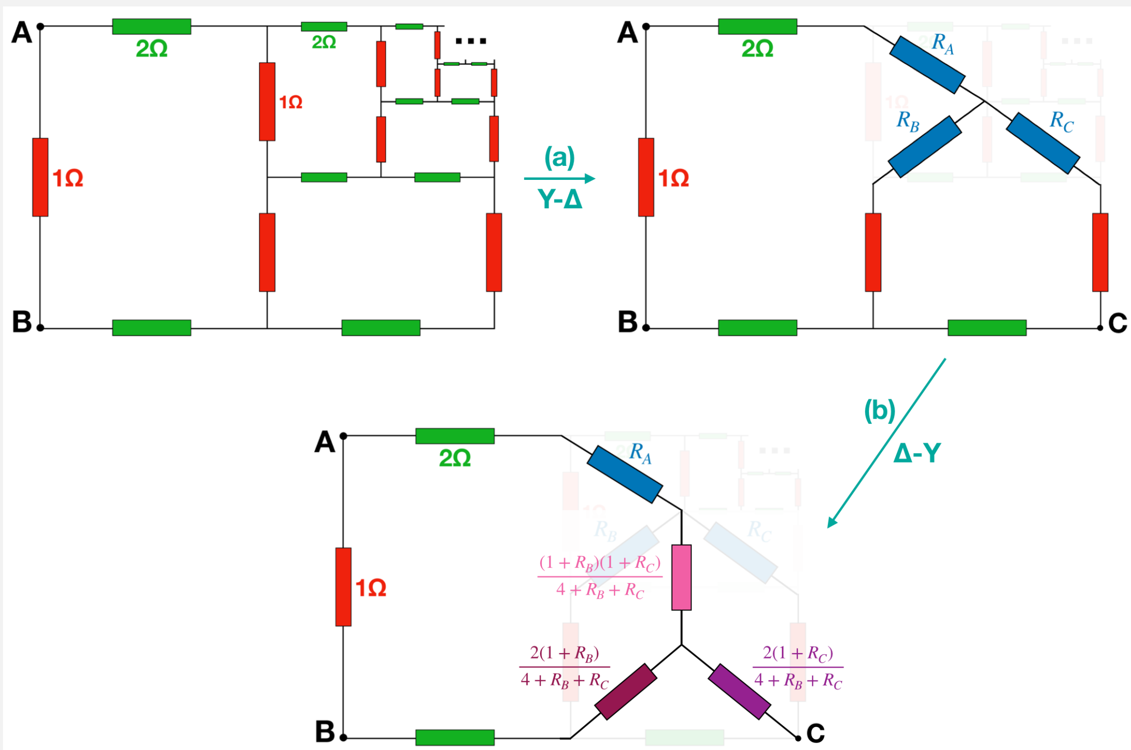
$$\epsilon(\lambda, T) = -173333(\lambda - 1000 \cdot 10^{-9}) + 0.366,$$

where λ is in meters. Numerically finding the maximum of $B(\lambda, T)$ with respect to λ (i.e. with a graphing calculator), we get the new peak wavelength to be $\lambda_{new} = 949$ nm, while the old peak wavelength (assuming a perfect blackbody) is $\lambda_{old} = 1035$ nm, and their percent difference (rounded to 1 significant digit) is 0.08. $\boxed{0.08}$.

27. Bioshock Infinite The equivalent resistance (in Ω) between points A and B of the following infinite resistance network made of 1Ω and 2Ω resistors is $\overline{0.abcdefgh...}$ in decimal form. Enter $\overline{efg}$ into the answer box (It should be an integer in the range of 0-999).



Solution 27: We consider two-steps: (a) Y-to-Δ transformation and (b) Δ-to-Y transformation on this resistance network (see attached figure), which R_A, R_B and R_C are unknowns to be determined (note these resistance values are always non-negative $R_A, R_B, R_C \geq 0$).



Stop at the transformed network after step (a), the equivalent resistance R_{AB} between points A and B can be calculated in two different ways:

$$R_{AB} = R_A + R_B = \frac{1}{\frac{1}{1} + \frac{1}{4+R_A+\frac{1}{\frac{1}{R_B}+\frac{1}{3+R_C}}}} \, . \tag{31}$$

Stop at the transformed network after step (b), the equivalent resistance R_{BC}between points B, C and R_{CA} between points C, A can also be calculated in two different ways:

$$R_{BC} = R_B + R_C = \frac{1}{\frac{1}{3+R_A+\frac{(1+R_B)(1+R_C)}{4+R_B+RC}} + \frac{1}{2+\frac{2(1+R_B)}{4+R_B+R_C}}} + \frac{2(1+R_C)}{4+R_B+R_C} \, , \tag{32}$$

$$R_{CA} = R_C + R_A = \frac{2(1+R_C)}{4+R_B+R_C} + \frac{1}{\frac{1}{2+R_A+\frac{(1+R_B)(1+R_C)}{4+R_B+RC}} + \frac{1}{3+\frac{2(1+R_B)}{4+R_B+R_C}}} \; . \tag{33}$$

Define the variable λ:

$$\lambda = R_A R_B + R_B R_C + 4R_A + 8R_B + 6R_C + 23 \geq 23 \; . \tag{34}$$

Expanding Eq. (31) gives:

$$R_A + R_B = \frac{(-4+\lambda) - R_B - R_C}{\lambda} \;\Rightarrow\; \lambda R_A + (1+\lambda)R_B + R_C = (-4+\lambda) \; , \tag{35}$$

while expanding Eq. (32) and Eq. (33) then dividing both their nominators and denominators with a non-zero value $(4 + R_B + R_C) \neq 0$ gives:

$$\begin{aligned} R_B + R_C &= \frac{(-48+4\lambda) - 4R_A - 16R_B - 4R_C}{\lambda} \\ &\Rightarrow\; 4R_A + (16+\lambda)R_B + (4+\lambda)R_C = (-48+4\lambda) \; , \end{aligned} \tag{36}$$

$$\begin{aligned} R_C + R_A &= \frac{(-72+5\lambda) - 4R_A - 25R_B - 9R_C}{\lambda} \\ &\Rightarrow\; (4+\lambda)R_A + 25R_B + (9+\lambda)R_C = (-72+5\lambda) \; . \end{aligned} \tag{37}$$

The system of three linear-equations with three unknowns, Eq. (35), Eq. (36), Eq. (37) can be solved by the ratios of 3×3 matrices' determinants as followed:

$$R_A = \frac{\det\begin{vmatrix} (-4+\lambda) & (1+\lambda) & 1 \\ (-48+4\lambda & (16+\lambda) & (4+\lambda) \\ (-72+5\lambda) & 25 & (9+\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{-40+120\lambda-24\lambda^2+\lambda^3}{(2+\lambda+2)(4+\lambda^2)} \; , \tag{38}$$

$$R_B = \frac{\det\begin{vmatrix} \lambda & (-4+\lambda) & 1 \\ 4 & (-48+4\lambda) & (4+\lambda) \\ (4+\lambda) & (-72+5\lambda) & (9+\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{2(-4-32\lambda+9\lambda^2)}{(2+\lambda+)(4+\lambda^2)} \; , \tag{39}$$

$$R_C = \frac{\det\begin{vmatrix} \lambda & (1+\lambda) & (-4+\lambda) \\ 4 & (16+\lambda) & (-48+4\lambda) \\ (4+\lambda) & 25 & (-72+5\lambda) \end{vmatrix}}{\det\begin{vmatrix} \lambda & (1+\lambda) & 1 \\ 4 & (16+\lambda) & (4+\lambda) \\ (4+\lambda) & 25 & (9+\lambda) \end{vmatrix}} = \frac{2(-12+52\lambda-39\lambda^2+2\lambda^3)}{(2+\lambda)(4+\lambda^2)} \; . \tag{40}$$

Plug Eq. (38), Eq. (39), Eq. (40) into Eq. (34), after an exhausting (but trivial) brute-force algebraic manipulation we arrive at:

$$\lambda = \frac{\lambda^2(220-365\lambda+55\lambda^2)}{(2+\lambda)^2(4+\lambda^2)} \;\Rightarrow\; 16 - 204\lambda + 364\lambda^2 - 51\lambda^3 + \lambda^4 = 0 \; . \tag{41}$$

This Eq. (41) is a quartic polynomal, which solutions have algebraic expressions (general formula is known, can be found in textbooks or Wikipedia). Since $\lambda \geq 23$, the physical solution is:

$$\lambda = \frac{1}{4}\left(51 + \sqrt{1177} + \sqrt{3714 + 102\sqrt{1177}}\right) . \tag{42}$$

From (31), replacing λ in Eq. (38) and (39) with the value given in Eq. (42):

$$\begin{aligned} R_{AB} = R_A + R_B &= \frac{-48 + 56\lambda - 6\lambda^2 + \lambda^3}{(2+\lambda)(4+\lambda^2)} \\ &= \frac{1}{356}\left(-482 - 8\sqrt{1177} + \sqrt{678070 + 12874\sqrt{1177}}\right) . \end{aligned} \tag{43}$$

With a handheld 8-digit calculator, it is possible with a good choice for order of arithmetic operations to obtain the numerical value with very high precision. Here's an example:

(1) Input 1177, $\sqrt{\ }$, $\times$, 12874, $+$, 678070, $\sqrt{\ }$ then memorize this value M.

(2) Input 1177, $\sqrt{\ }$, $\times$, 8, $+$, 482, $-$, M, $/$, 35.6 then the screen will show -8.4752823.

Get rid of the minus sign and move the decimal sign forward 1-digit, the numerical value for R_{AB} is approximately 0.847528233Ω. We then can use 282 as the answer for this physics puzzle, which is indeed in great agreement with better calculators.

$\boxed{282}$

* This puzzle was created with helps from Tuan K. Do.

28. Magnetic Ball A uniform spherical metallic ball of mass m, resistivity ρ, and radius R is kept on a smooth friction-less horizontal ground. A horizontal uniform, constant magnetic field B exists in the space parallel to the surface of ground. The ball was suddenly given an impulse perpendicular to magnetic field such that ball begin to move with velocity v without losing the contact with ground. Find the time in seconds required to reduce its velocity by half.

Numerical Quantities: $m = 2$ kg, $4\pi\epsilon_0 R^3 B^2 = 3$ kg, $\rho = 10^9\Omega$m, $v = \pi$ m/s.

Solution 28: WLOG assuming magnetic field to be into the plane (negative z-axis) and velocity of the block along x-axis. Let any time t ball has velocity v and surface charge $\sigma \cos\theta$, where θ is measured from Y-axis. As the ball is moving it will also generate and electric field E=vB along positive Y-axis. Which will be opposed by the electric field of ball. Also we know the electric filed generated by the ball is $\dfrac{\sigma}{3\epsilon_0}$ in negative Z-axis. As this charge distribution will arise form a vertical electric field and the subsequent induced charges will also produce only a vertical electric field only thus our assumption about charge distribution and net electric field must be true.

Now at any point on the surface of the sphere rate of increase in surface charge density is given by-

$$JdA\cos\theta = \frac{d(\sigma\cos\theta)}{dt}dA$$
$$J = \frac{d\sigma}{dt}$$
$$E - \frac{\sigma}{3\epsilon_0} = \rho\frac{d\sigma}{dt}$$
$$vB - \frac{\sigma}{3\epsilon_0} = \rho\frac{d\sigma}{dt} \tag{44}$$

As the magnetic field is uniform to calculate force path of the current will not matter. Hence assuming it to be straight line between two points located at θ and $-\theta$. So the force can be written as -

$$dF = B \times dI \times l$$
$$dF = B(2\pi(R\sin\theta) \times Rd\theta \times J)(2R\cos(\theta))$$
$$F = 4\pi R^3 B\frac{d\sigma}{dt}\int_0^{\pi/2}\sin\theta\cos^2\theta d\theta$$
$$F = \frac{4}{3}\pi R^3 B\frac{d\sigma}{dt} \tag{45}$$

Now writing force equation on the sphere we have

$$F = -m\frac{dv}{dt}$$
$$\frac{4}{3}\pi R^3 B\frac{d\sigma}{\cancel{dt}} = -m\frac{dv}{\cancel{dt}}$$
$$\frac{4}{3}\pi R^3 B\int_0^{\sigma}d\sigma = -m\int_{v_0}^{v}dv$$
$$\frac{4}{3}\pi R^3 B\sigma = mv_0 - mv \tag{46}$$

Solving equation 1 and 3 gives us

$$\left(m + 4\pi R^3 B^2\epsilon_0\right)v - mv_0 = -3m\rho\epsilon_0\frac{dv}{dt}$$

Integrating it from v_0 to $\frac{v_0}{2}$ gives

$$t = \frac{3m\rho\epsilon_0}{(m + 4\pi R^3 B^2\epsilon_0)}\ln\left(\frac{8\pi R^3 B^2\epsilon_0}{4\pi R^3 B^2\epsilon_0 - m}\right) \tag{47}$$
$$t = \frac{6\ln(6)}{5}\rho\epsilon_0 = 0.019 \text{ s} \tag{48}$$

$\boxed{0.019 \text{ s}}$

For the following two problems, this information applies. Assume $g = 9.8\text{m/s}^2$. On a balcony, a child holds a spherical balloon of radius 15 cm. Upon throwing it downwards with a velocity of 4.2 m/s, the balloon starts magically expanding, its radius increasing at a constant rate of 35 cm/s. Another child, standing on the ground, is holding a hula hoop, 4 m below the point where the center of the balloon was released.

29. Magical Balloon 1 If the minimum radius of the hoop such that the balloon falls completely through the hula hoop without touching it is r, compute the difference between r and the largest multiple of 5cm less than or equal to r. Answer in centimeters; your answer should be in the range $[0, 5)$.

30. Magical Balloon 2 Consider the horizontal plane passing through the center of the balloon at the start. If the total volume above this plane that the balloon falls through after it is thrown downwards is V, compute the difference between V and half the original volume of the balloon. Answer in milliliters; your answer should be nonnegative.

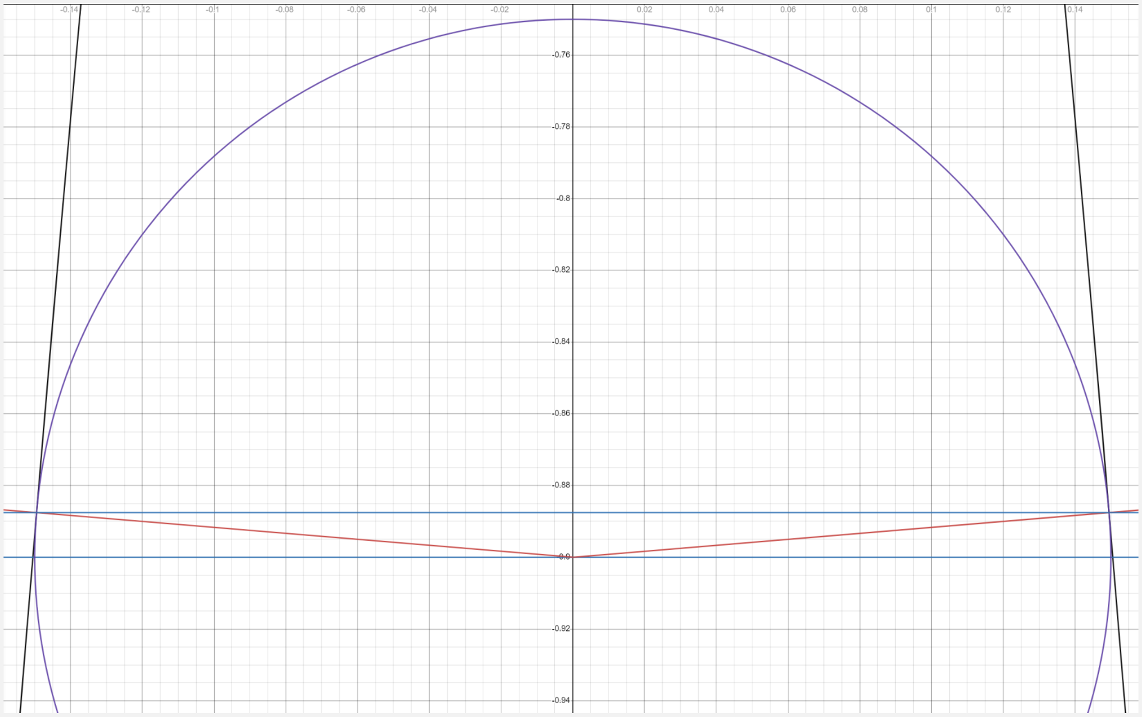
Note that when refer to the "volume an object falls through", it refers to the volume of the union of all points in space which the object occupies as it falls.

Solution 29: The first key insight comes by noting that the numbers in the problem are carefully chosen. If we go backwards in time, the balloon had radius 0 at a time $t_0 = -\frac{3}{7}$sec. However, then the balloon's downwards velocity at that point would have been $v_0 = 4.2 - gt = 0$m/s - also zero velocity! Therefore, we can consider the balloon as having been released from rest at a height $4 + \frac{v^2}{2g} = 4.9$m above the lower hula-hoop.
If we consider the balloon as a whole, it both expands and falls, making the overall volume occupied by it over time difficult to calculate. The next key insight comes from thinking about individual points on the balloon, relative to the center of mass. The center of the balloon falls under gravity; relative to the center of mass, a point on the balloon travels away from it at a constant rate. However, the motion of this point is identical to a projectile, launched from the starting point at a speed equal to the expansion rate, in the direction of expansion.
Therefore, the needed radius of the bottom hula hoop is the maximum distance a projectile launched at 35cm/s can travel horizontally, before falling 4.9m. This is sufficient to solve the problem, getting a distance of 35.0223cm and therefore an answer of 0.0223, but an easier solve can be obtained by making more insights, described below.

Solution 30: After turning the problem into a projectile distance problem, the third key insight comes from realizing that the distance to the edge of the volume fell through at any angle of elevation or depression is the same as the maximum distance that a projectile can travel up or down a slope of the same angle. Through either intuition or polar coordinate bashing, we can determine the shape of the surface, which is a paraboloid of revolution.
The easiest way to parametrize the paraboloid is with the zero-velocity balloon at $(0, 0)$. Three points on the parabola can easily be found by considering the maximum horizontal projectile distance and the maximum vertical projectile distance: $\left(0, \frac{v^2}{2g} = 0.00625\right)$ and $\left(\frac{v^2}{g} = 0.0125, 0\right)$.
Therefore, we get the parametrization of the paraboloid:

$$f(x) = \frac{v^2}{2g} - \frac{g}{2v^2}x^2 = 0.00625 - 40x^2$$

By finding x for $y = -4.9$m, this immediately provides a faster solution to the previous question.
Plotting the initial location of the ball in purple, as well as the paraboloid in black:



Notice, however, that the balloon does not exactly meet the edge of the paraboloid at the starting height (the lower blue line) - the intersection is actually a bit above, and the paraboloid is actually a bit wider at $y = -0.9$m. This slight discrepancy creates a difference in volume. We now seek to compute this difference.

The balloon falls at a rate of 4.2m/s at this point, while expanding at 35cm/s radially outwards, and this radially outwards vector is perpendicular to the slope of the paraboloid at the point where the paraboloid and balloon touch. Solving the triangle, the red line has a slope of $\frac{1}{\sqrt{143}}$. This line intersects with the balloon edge (and the paraboloid) at $y = -0.8875$m (upper blue line). It can be checked that this point is on both the paraboloid and the balloon.

We know the volume of a paraboloid is $V = r^2 h \int_0^1 2\pi r(1-r^2)dr = \frac{\pi}{2} r^2 h$. To compute the volume of the paraboloid between these two lines, we take the difference between the volume above the lower and upper lines. The upper line gives:

$$V_1 = \frac{\pi}{2}\frac{2v^2}{g}\left(0.8875 + \frac{v^2}{2g}\right)^2 = 31368.3731\text{mL}$$

While the lower line gives:

$$\frac{\pi}{2}\frac{2v^2}{g}\left(.9 + \frac{v^2}{2g}\right)^2 = 32251.9461\text{mL}$$

The difference between the two gives the volume between, which is 883.5729mL.

We now compute the volume of the original balloon between these two lines, which we split up into a portion of the sphere (below the red line) and a cone (above the red line). The proportion of the sphere is given by a solid angle integral:

$$f = \frac{1}{2}\int_0^{\sin^{-1} .35/4.2} \cos\theta d\theta = \frac{1}{24}$$

And the volume:

$$V_{\text{sphere}} = \frac{4}{3}\pi\left(.15\right)^3 f = 589.0486\text{mL}$$

While the volume of the cone is given by:

$$V_{\text{cone}} = \frac{1}{3}\pi\frac{2v^2}{g}\left(0.8875 + \frac{v^2}{2g}\right)(0.9 - 0.8875) = 292.4790\text{mL}$$

Summing gives 881.5276mL.
Finally, taking the difference of the two volumes gives our final result, 2.0453mL.

31. Hydrogen Magnetism In quantum mechanics, when calculating the interaction between the electron with the proton in a hydrogen atom, it is necessary to compute the following volume integral (over all space):

$$\mathbf{I} = \int \mathbf{B}(\mathbf{r})|\Psi(\mathbf{r})|^2 dV$$

where $\Psi(\mathbf{r})$ is the spatial wavefunction of the electron as a function of position $\mathbf{r}$ and $\mathbf{B}(\mathbf{r})$ is the (boldface denotes vector) magnetic field produced by the proton at position $\mathbf{r}$. Suppose the proton is located at the origin and it acts like a finite-sized magnetic dipole (but much smaller than a_0) with dipole moment $\mu_p = 1.41 \times 10^{-26}$ J/T. Let the hydrogen atom be in the ground state, meaning $\Psi(\mathbf{r}) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$, where $a_0 = 5.29 \times 10^{-11}$ m is the Bohr radius. Evaluate the magnitude of the integral $|\mathbf{I}|$ (in SI units).

Solution 31:
First, note that the result of the integral will be a vector in the direction the dipole is pointing, call it the z-direction. Thus we can replace $\mathbf{B}$ in the integral with B_z. Note that for any $R > 0$, the integral over the space outside the sphere of radius R is 0. To show this, since $|\Psi|$ is exponentially decaying, we only need to show that the integral over a spherical shell is 0. To show this, we can show that the integral of $\mathbf{B}$ inside a sphere of radius R is independent of R. Indeed, this quickly follows from dimensional analysis (the only relevant quantities are μ_0, μ_p, and R, and one can check that $\mu_0\mu_p$ already gives the right dimensions, and there is no dimensionless combination of these 3 quantities. In fact, we will actually compute this integral at the end.)
Now, it suffices to compute the integral of $\mathbf{B}|\Psi|^2$ inside the sphere. Since R was arbitrary, we can make it very small, much smaller than a_0. Then we can replace $|\Psi(\mathbf{r})|^2$ with $|\Psi(0)|^2 = \frac{1}{\pi a_0^3}$, a constant that can be factored out. The problem reduces to computing the integral of $\mathbf{B}$ inside a sphere of radius R.
We can compute this integral by splitting the sphere up into many thin discs, all perpendicular to the z axis. We have to add up the $\mathbf{B}$ field integrated over the volume of each disc, which is equivalent to the magnetic flux through the disc times the thickness of the disc. The magnetic flux through each disc can be calculated using the mutual inductance reciprocity theorem. Suppose a current I goes around the boundary of the disc (a ring) with radius r. Then the mutual inductance M between the ring and the dipole is given by the flux through the dipole divided by I:

$$M = \frac{B * A}{I}$$

where B is the magnetic field produced by the ring's current at the dipole's position, and A is the area of the dipole. The dipole itself carries current $i = \frac{\mu_p}{A}$, so the flux through the ring is given by

$$\Phi = M * i = \frac{BiA}{I} = \frac{\mu_p B}{I}$$

where $B = \frac{\mu_0 I}{4\pi} * \frac{2\pi r^2}{(r^2+z^2)^{\frac{3}{2}}} = \frac{\mu_0 I r^2}{2(r^2+z^2)^{\frac{3}{2}}}$, where z is the z coordinate of the ring. Using $r = R\sin\theta$ and $z = R\cos\theta$, we obtain

$$\Phi(\theta) = \frac{\mu_0 \mu_p r^2}{2(r^2+z^2)^{\frac{3}{2}}} = \frac{\mu_0 \mu_p \sin^2\theta}{2R}$$

Finally, we integrate over the thickness of the disc $R\sin\theta d\theta$ to get :

$$\int_0^\pi \Phi(\theta) R\sin\theta d\theta = \frac{1}{2}\mu_0\mu_p \int_0^\pi \sin^3\theta d\theta = \frac{2}{3}\mu_0\mu_p$$

Thus, $|\mathbf{I}| = \frac{2}{3}\mu_0\mu_p * \frac{1}{\pi a_0^3} = \frac{2\mu_0\mu_p}{3\pi a_0^3} = 0.0254$ T.

32. Relativistic Collision Zed is trying to model the repulsive interaction between 2 objects, A and B (with masses m_A and m_B, respectively), in a relativistic setting. He knows that in relativity, forces cannot act at a distance, so he models the repulsive force with a small particle of mass m that bounces elastically between A and B. Throughout this problem, assume everything moves on the x-axis. Suppose that initially, A and B have positions and velocities x_A, v_A and x_B, v_B, respectively, where $x_A < x_B$ and $v_A > v_B$. The particle has an initial (relativistic) speed v.

For simplicity, assume that the system has no total momentum. You may also assume that $v_A, v_B \ll v$, and that $p_m \ll p_A, p_B$, where p_m, p_A, p_B are the momenta of the particle, A, and B, respectively. Do NOT assume $v \ll c$, where c is the speed of light.

Find the position (in m) of A when its velocity is 0, given that $m_A = 1$ kg, $m_B = 2$ kg, $v_A = 0.001c$, $m = 1 \times 10^{-6}$ kg, $v = 0.6c$, $x_A = 0$ m, $x_B = 1000$ m.

Note: Answers will be tolerated within 0.5%, unlike other problems.

Solution 32:
Since total momentum is 0, we have $m_A v_A + m_B v_B = 0$, so $v_B = -0.0005c$ By conservation of energy:

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 + \gamma_0 mc^2 = \gamma mc^2$$

where we define $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ to correspond to the final state of the particle and γ_0 to the initial state. This equation allows us to solve for γ. Since the speed of the particle is much larger than the speed of the masses A and B, we can imagine the particle moving in a infinite well potential where the walls are slowly moving. Applying the adiabatic theorem, we get that the adiabatic invariant px is conserved, where p is the particle's momentum, and x is the distance between A and B. Thus, $\gamma v x$ is conserved, so

$$\gamma v x = \gamma_0 v_0 (x_B - x_A)$$

We can solve for x, since we know γ and v. Finally, we realize that the center of mass stays at $x_c m = \frac{m_A x_A + m_B x_B}{m_A + m_B}$, so the final position of A is simply

$$x_c m - \frac{m_B}{m_A + m_B} x = 378 \text{ m}$$

33. Microscope Consider an optical system consisting of two thin lenses sharing the same optical axis. When a cuboid with a side parallel to the optical axis is placed to the left of the left lens, its final image formed by the optical system is also a cuboid but with 500 times the original volume. Assume the two lenses are 10 cm apart and such a cuboid of volume 1 cm^3 is placed such that its right face is 2 cm to the left of the left lens. What's the maximum possible volume of the intermediate image (i.e., image formed by just the left lens) of the cuboid? Answer in cm^3.

Solution 33:
First, note that the two lenses share a focal point. Here's why. For any cuboid with four edges parallel to the optical axis, consider the four parallel rays of light that these four edges lie on. The intermediate images formed by the left lens of these four edges lie on these same light rays after they've passed through the left lens, and the final images of the edges (images formed by the right lens of the intermediate images) lie on these same light rays after they've also passed through the right lens. Since the initial rays were parallel and the same goes for the final rays, the intermediate rays intersect at a focal point of both the left lens and the right lens at the same time.
Now, let f, f' be the focal lengths of the left and right lenses, respectively. Although they can be any nonzero real number, we will WLOG assume that they are positive. (The following derivation will still hold with either f or f' negative, as long as we have the right sign conventions for all the variables.)
For a point at a distance x_1 to the left of F, its image is at a distance $x_2' = -\left(\frac{f'}{f}\right)^2 x_1$ to the right of F'. This follows from applying Newton's formula twice:

$$x_2' = \frac{f'^2}{x_1'} = -\frac{f'^2}{x_2} = -\frac{f'^2}{f^2}x_1.$$

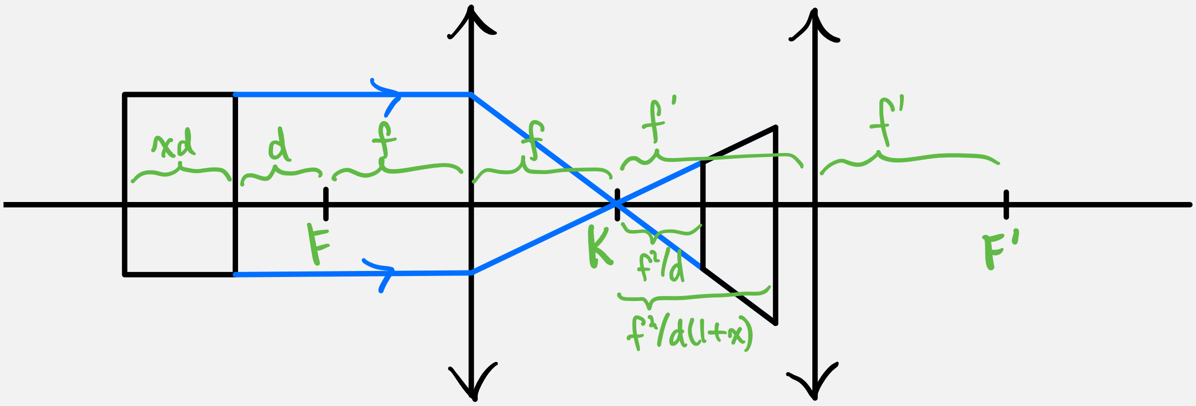
Thus, the optical system magnifies horizontal distances by $\left(\frac{f'}{f}\right)^2$.
On the other hand, for a point at height h_1 (relative to the optical axis), consider a horizontal light ray through the point. Then the final light ray (after it passes through both lenses) is at a height of

$$h_2 = -\frac{f'}{f}h_1,$$

which is the height of the final image of the point. Hence, the optical system magnifies transverse distances (i.e., distances perpendicular to the optical axis) by $\frac{f'}{f}$.
The two results above imply that volumes are magnified by

$$\left(\frac{f'}{f}\right)^2\left(\frac{f'}{f}\right)^2 = \left(\frac{f'}{f}\right)^4.$$

(The second factor is squared because there are two transverse dimensions.) Given that volumes are magnified by 500 times, we obtain $\frac{f'}{f} = \pm 500^{1/4}$.
We now look at a cuboid with volume V whose right face is at a distance d to the left of F. Let it have width xd and transverse cross-sectional area $A = V/xd$. The intermediate image is a frustrum that results from truncating a pyramid with vertex located at K.



By Newton's formula, the bases of the frustrum are at distances $\frac{f^2}{d}$ and $\frac{f^2}{d(1+x)}$ to the right of K, and they have areas

$$\left(\frac{\frac{f^2}{d}}{f}\right)^2 A = \frac{Vf^2}{xd^3} \quad \text{and} \quad \left(\frac{\frac{f^2}{d(1+x)}}{f}\right)^2 A = \frac{Vf^2}{x(1+x)^2d^3}.$$

Thus, the volume of the frustrum is

$$\frac{1}{3}\left(\frac{f^2}{d}\frac{Vf^2}{xd^3} - \frac{f^2}{d(1+x)}\frac{Vf^2}{x(1+x)^2d^3}\right) = \frac{1}{3}\frac{Vf^4}{xd^4}\left(1 - \frac{1}{(1+x)^3}\right) \leq \frac{1}{3}\frac{Vf^4}{xd^4}(1-(1-3x)) = \frac{Vf^4}{d^4},$$

where equality is approached as $x \to 0$.
Since $f + f' = 10$ cm and $\frac{f'}{f} = \pm 500^{1/4} \approx \pm 4.7287$, either $f = 1.7456$ cm and $d = 2\text{ cm} - f = 0.2544$ cm, which gives $V_{max} = \frac{Vf^4}{d^4} = 2216\text{ cm}^3$, or $f = -2.6819$ cm and $d = 2\text{ cm} - f = 4.6819$ cm, which gives $V_{max} = 0.1077\text{ cm}^3$. The former is larger, so the answer is 2216 cm^3.

34. Resistor grid Consider an infinite square grid of equal resistors where the nodes are exactly the lattice points in the 2D Cartesian plane. A current $I = 2.7$ A enters the grid at the origin $(0,0)$. Find the current in Amps through the resistor connecting the nodes $(N,0)$ and $(N,1)$, where $N = 38$ can be assumed to be much larger than 1.

Solution 34:
WLOG, let each resistor have unit resistance.
Kirchoff's current law says that the total current entering the node $(x,y) \neq (0,0)$ is zero:

$$[U(x+1,y) - U(x,y)] + [U(x-1,y) - U(x,y)] + [U(x,y+1) - U(x,y)] + [U(x,y-1) - U(x,y)] = 0$$
$$[U(x+1,y) - 2U(x,y) + U(x-1,y)] + [U(x,y+1) - 2U(x,y) + U(x,y-1)] = 0.$$

This is an approximation of the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

which is Laplace's equation $\nabla^2 U = 0$ in 2D. Given $U \to 0$ as $r \to \infty$, this implies that the potential

field approximates that of a point charge at O in 2D. This approximation is valid far from the origin where changes in U over unit length are small.
The electric field corresponding to this potential field is

$$\mathbf{E} = -\nabla U = (-\partial_x U, -\partial_y U) \approx (U(x-1,y) - U(x,y), U(x,y-1) - U(x,y)) = (i_x(x,y), i_y(x,y))$$

far from the origin, where $i_x(x,y), i_y(x,y)$ are the horizontal and vertical currents passing through node (x,y). (Note that the current horizontal current is different to the left vs. to the right of the node, and similarly for the vertical current, but the difference is negligible for $N \gg 1$.)
The current $i = \sqrt{i_x^2 + i_y^2}$ at a distance $r \gg 1$ away from the origin is given by Gauss's law:

$$2\pi r i(r) \approx I,$$

so at $(N,0)$ we have

$$i_x(N,0) = i(N) \approx \frac{I}{2\pi N}.$$

The difference between the entering and exiting horizontal currents at $(N,0)$ is approximately

$$-\partial_x i_x(N,0) = \frac{I}{2\pi N^2}.$$

This difference is directed equally into the vertical resistors adjacent to $(N,0)$, so the final answer is

$$\frac{I}{4\pi N^2} = 1.488 \times 10^{-4} \text{ A}.$$

35. Strange Gas Suppose we have a non-ideal gas, and in a certain volume range and temperature range, it is found to satisfy the state relation

$$p = AV^\alpha T^\beta$$

where A is a constant, $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{2}$, and the other variables have their usual meanings. Throughout the problem, we will assume to be always in that volume and temperature range.

Assume that $\gamma = \frac{C_p}{C_V}$ is found to be constant for this gas (γ is independent of the state of the gas), where C_p and C_v are the heat capacities at constant pressure and volume, respectively. What is the minimum possible value for γ?

Solution 35:
We claim that the conditions given uniquely determine γ.
The fundamental thermodynamic relation gives:

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$$

So

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

where we have used a Maxwell relation.

$$\mathrm{d}U = \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial U}{\partial V}\right)_T \mathrm{d}V = C_V \mathrm{d}T + \left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right)\mathrm{d}V$$

We have

$$C_p = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = C_V + T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p$$

Rearranging, gives

$$C_V = \frac{T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p}{\gamma - 1}$$

From the symmetry of mixed second partial derivatives, we know

$$\left(\frac{\partial C_V}{\partial V}\right)_T = \left(\frac{\partial^2 U}{\partial T \partial V}\right) = \left(\frac{\partial}{\partial T}\left(\frac{\partial U}{\partial V}\right)_T\right)_V = \frac{\partial}{\partial T}\left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right) = \left(\frac{\partial^2 p}{\partial T^2}\right)_V$$

Plugging our expression for C_V into here, and plugging in the equation of state, we can solve for γ to get

$$\gamma = \frac{\alpha + \beta}{\alpha(1 - \beta)} = \frac{7}{4}$$