2021 Online Physics Olympiad: Open Contest Solutions

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

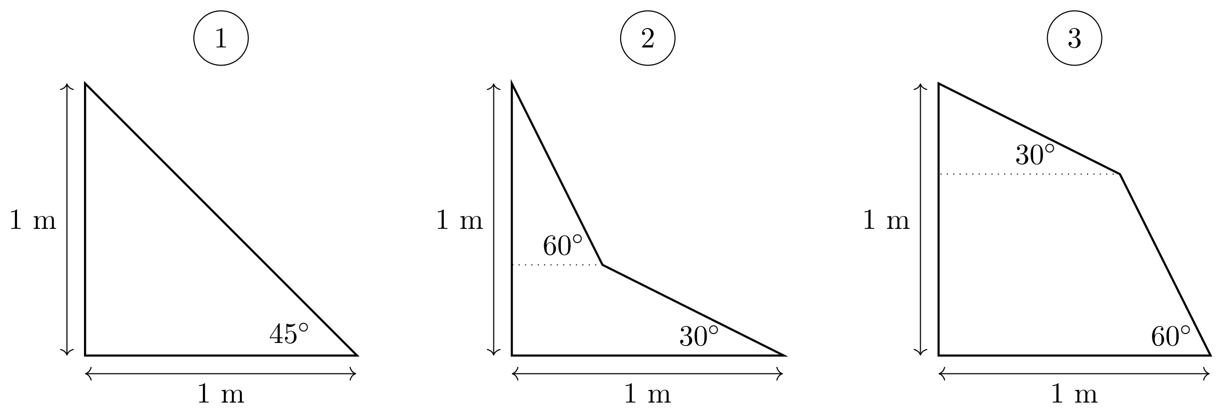
- Use $g = 9.81$ m/s^2 in this contest. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts that you take to solve a problem as well as the number of teams who solve it. **Note:** Unlike last year, the time it takes will no longer be a factor.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed or allowed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on June 6, 2021 at 11:59 PM UTC**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol · K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, $1 \text{ eV} = 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11} \text{ (N} \cdot \text{m}^2\text{)/kg}^2$$
- Acceleration due to gravity, $g = 9.81$ m/s^2
- 1 unified atomic mass unit,
$$1 \text{ u} = 1.66 \cdot 10^{-27} \text{ kg} = 931 \text{ MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34} \text{ J} \cdot \text{s} = 4.41 \cdot 10^{-15} \text{ eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9 \text{ (N} \cdot \text{m}^2\text{)/C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7} \text{ T} \cdot \text{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \text{ (T} \cdot \text{m)/A}$$
- 1 atmospheric pressure,
$$1 \text{ atm} = 1.01 \cdot 10^5 \text{ N/m}^2 = 1.01 \cdot 10^5 \text{ Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m · K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8} \text{ W/m}^2\text{/K}^4$$

Problems

1. Fastest Path A small toy car rolls down three ramps with the same height and horizontal length, but different shapes, starting from rest. The car stays in contact with the ramp at all times and no energy is lost. Order the ramps from the fastest to slowest time it takes for the toy car to drop the full 1 m. For example, if ramp 1 is the fastest and ramp 3 is the slowest, then enter 123 as your answer choice.



Solution: One can solve this by finding the time it takes for each ramp. For ramp 1:

$$\sqrt{2} = \frac{1}{2} g \sin(45°) t^2$$
$$\implies t = 0.639 \text{ s}$$

For ramps 2, let the length of the dashed region be x. Then:

$$x + x/\tan(30°) = 1 \implies x = 0.366 \text{ m} \quad (1)$$

Due to symmetry, both the steep and shallow regions of both ramps 2 and 3 have a length of $x/\cos(60°) = 0.732$ m. This results in a time for ramp 2 as:

$$0.732 = \frac{1}{2} g \sin(60°) t_1^2 \implies t_1 = 0.415 \text{ s}$$
$$0.732 = \sqrt{2g(1-x)} t_2 + \frac{1}{2} g \sin(30°) t_2^2 \implies t_2 = 0.184 \text{ s}$$

for a total time of $t_1 + t_2 = 0.599$ s. For ramp 3,

$$0.732 = \frac{1}{2} g \sin(30°) t_1^2 \implies t_1 = 0.546 \text{ s}$$
$$0.732 = \sqrt{2g(1-x)} t_2 + \frac{1}{2} g \sin(60°) t_2^2 \implies t_2 = 0.206 \text{ s}$$

which gives a total time of $t_1 + t_2 = 0.752$ s. From fastest to slowest, the answer becomes 213. Note that this answer is easily guessable via intuition.

$\boxed{213}$

2. Resistor Puzzle What is the smallest number of 1Ω resistors needed such that when arranged in a certain arrangement involving only series and parallel connections, that the equivalent resistance is $\frac{7}{6}\Omega$?

Solution: We can write:

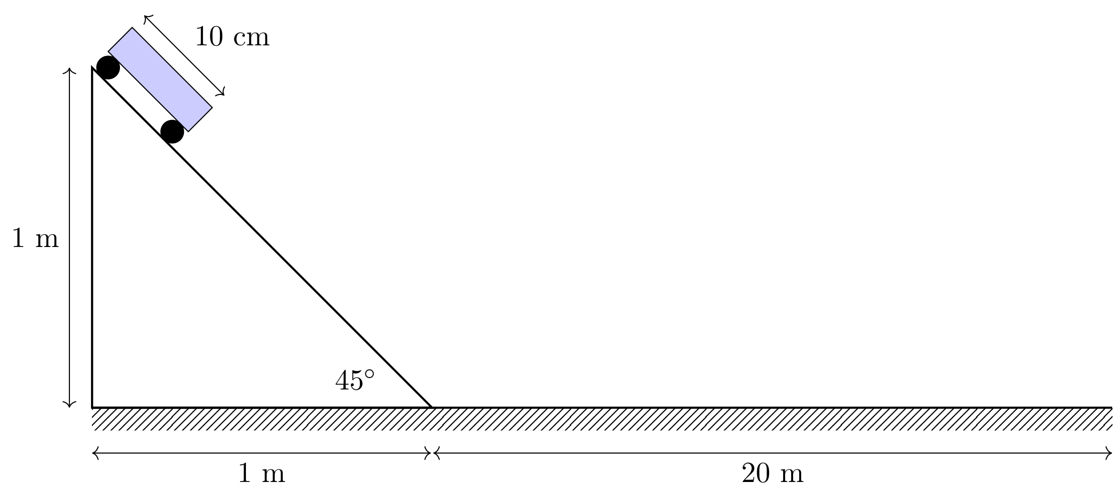
$$\frac{7}{6} = \frac{1}{2} + \frac{2}{3} \tag{2}$$

It takes two resistors (connected in parallel) to create a $\frac{1}{2}\Omega$ resistor. If we write $\frac{2}{3} = \frac{2\cdot 1}{2+1}$, then it takes three resistors to create a $\frac{2}{3}\Omega$ (this is accomplished by connecting a 2 resistor in parallel with a 1 resistor.

Combining the $\frac{1}{2}\Omega$ element in series with the $\frac{2}{3}\Omega$ element gives us our desired amount. To prove this is the minimum, we can easily check all possible combinations using 4 or fewer resistors.

$\boxed{5}$

3. Derby Race In a typical derby race, cars start at the top of a ramp, accelerate downwards, and race on a flat track, and are always set-up in the configuration shown below.



A common technique is to change the location of the center of mass of the car to gain an advantage. Alice ensures the center of mass of her car is at the rear and Bob puts the center of mass of his car at the very front. Otherwise, their cars are exactly the same. Each car's time is defined as the time from when the car is placed on the top of the ramp to when the front of the car reaches the end of the flat track. At the competition, Alice's car beat Bob's. What is the ratio of Bob's car's time and Alice's car's time?

Assume that the wheels are small and light compared to the car body, neglect air resistance, and the height of the cars are small compared to the height of the ramp. In addition, neglect all energy losses during the race and the time it takes to turn onto the horizontal surface from the ramp. Express your answer as a decimal greater than 1.

Solution: The speed at which Alice's car hits the ground is given by $v_A = \sqrt{2g(1)} = 4.43$ m/s from energy conservation. The speed at which Bob's car hits the ground is given by $v_B = \sqrt{2g(1 - 0.1\sin(45^\circ)} = 4.27$ m/s since its center of mass is at the very front of the car.

The time in which Alice and Bob's cars are going down the ramp (and wheels are both on the ramp) is given by:

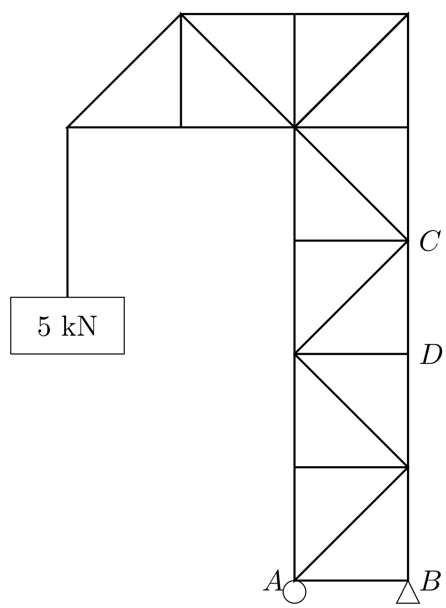
$$\sqrt{2} - 0.1 = \frac{1}{2} g \sin(45°) t^2 \implies t = 0.616 \text{ s} \tag{3}$$

And the time that Alice's car takes on the flat path (and wheels are both on the floor) is given by $t_A = \frac{20}{v_A} = 4.51$, and for Bob, $t_B = \frac{20}{v_B} = 4.68$.
We ignore the time it takes for the car to transition. The justification behind this is that this time would be on the order of magnitude of $\frac{0.1}{v_A} = 0.02$ s $\ll t_A$. In fact, it is smaller than the 1% margin allowed (so teams who did not read the clarification would still have gotten the correct answer).
The ratio asked for was therefore:

$$\frac{0.616 + t_B}{0.616 + t_A} \tag{4}$$

$\boxed{1.03}$

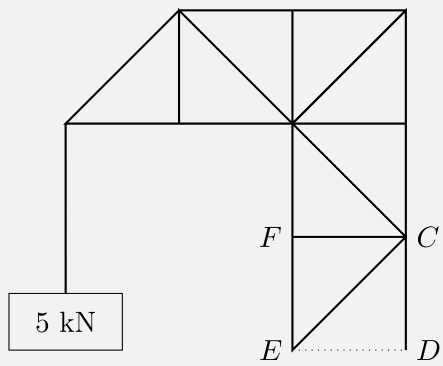
4. Crane A simple crane is shown in the below diagram, consisted of light rods with length 1 m and $\sqrt{2}$ m. The end of the crane is supporting a 5 kN object. Point B is known as a "pin." It is attached to the main body and can exert both a vertical and horizontal force. Point A is known as a "roller" and can only exert vertical forces. Rods can only be in pure compression or pure tension.



In kN, what is the force experienced by the rod CD? Express a positive number if the member is in tension and a negative number if it is in compression.

Solution: One naive method (though perfectly valid) is to solve for each member individually, starting from the two rods that connect to the 5kN weight. At each joint, we can write out force equilibrium equations in the vertical and horizontal directions, and solve a system of linaer equations to get the force in CD.

Instead, we can solve for this force in one line. Consider a horizontal slice right above point D.



Since the net force of this sub-element is still zero, we can do a force balance. The only *external* forces acting on this system is EF, EC, CD, and the 5kN weight. If we do a torque balance about E, we get:

$$5(2L) = CD(L) \tag{5}$$

where L is the length of the rod. This immediately gives $CD = 10\text{kN}$.

$\boxed{10\text{kN}}$

5. Coaxial Cable A coaxial cable is cylindrically symmetric and consists of a solid inner cylinder of radius $a = 2$ cm and an outer cylindrical shell of inner radius $b = 5$ cm and outer radius $c = 7$ cm. A uniformly distributed current of total magnitude $I = 5$ A is flowing in the inner cylinder and a uniformly distributed current of the same magnitude but opposite direction flows in the outer shell. Find the magnitude $B(r)$ of the magnetic field B as a function of distance r from the axis of the cable. As the final result, submit $\int_0^\infty B(r)\mathrm{d}r$. In case this is infinite, submit 42.

Solution: Ampere's law $\int B \cdot \mathrm{d}l = \mu_0 I$ is all we need. For every point on the wire, we can write the magnetic field as a function of the distance from its center r. Thus,

$$B(r) = \begin{cases} \frac{5\mu_0 r}{8\pi} & r \leq 2 \\ \frac{5\mu_0}{2\pi r} & 2 < r < 5 \\ \frac{5\mu_0(-r^2+49)}{48\pi r} & 5 \leq r \leq 7 \\ 0 & r > 7 \end{cases}$$

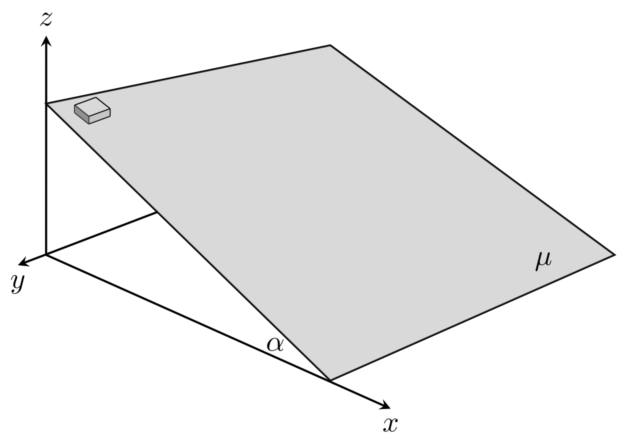
Now we just sum each integral from each interval, or in othe words

$$\int_0^\infty B(r)\mathrm{d}r = \int_0^2 B(r)\mathrm{d}r + \int_2^5 B(r)\mathrm{d}r + \int_5^7 B(r)\mathrm{d}r.$$

This is now straightforward integration.

$\boxed{1.6 \times 10^{-8}}$

6. Magnetic Block A small block of mass m and charge Q is placed at rest on an inclined plane with a slope $\alpha = 40°$. The coefficient of friction between them is $\mu = 0.3$. A homogenous magnetic field of magnitude B_0 is applied perpendicular to the slope. The speed of the block after a very long time is given by $v = \beta \frac{mg}{QB_0}$. Determine β. Do not neglect the effects of gravity.



Solution: Create a free body diagram. The direction of the magnetic field (into or out of the page) does not matter as we only need to know the magnitude of the terminal velocity. In dynamic equilibrium, we have three forces along the plane: the component of gravity along the plane, friction, and the magnetic force. The component of gravity along the plane is $mg \sin\alpha$. Friction has the magnitude $f = \mu mg \cos\alpha$. The magnetic force has magnitude $F_B = QvB_0$. At the terminal velocity, these forces are balanced. To finish, note that friction and the magnetic force are perpendicular so the magnitude of their vector sum is equal to the component of gravity along the plane. By the Pythagorean Theorem,

$$F_B^2 + f^2 = (mg \sin\alpha)^2 \implies v = \frac{mg}{QB_0}\sqrt{\sin^2\alpha - \mu^2 \cos^2\alpha}.$$

$\boxed{0.6}$

7. Thermal Train A train of length 100 m and mass 10^5 kg is travelling at 20 m/s along a straight track. The driver engages the brakes and the train starts deccelerating at a constant rate, coming to a stop after travelling a distance $d = 2000$ m. As the train decelerates, energy released as heat from the brakes goes into the tracks, which have a linear heat capacity of $5000 \mathrm{J\,m^{-1}\,K^{-1}}$. Assume the rate of heat generation and transfer is uniform across the length of the train at any given moment.

If the tracks start at an ambient temperature of 20°C, there is a function $T(x)$ that describes the temperature (in Celsius) of the tracks at each point x, where the rear of where the train starts is at $x = 0$. Assume (unrealistically) that 100% of the original kinetic energy of the train is transferred to the tracks (the train does not absorb any energy), that there is no conduction of heat along the tracks, and that heat transfer between the tracks and the surroundings is negligible.

Compute $T(20) + T(500) + T(2021)$ in degrees celsius.

Solution: Consider a small element of the tracks at position x with width $\mathrm{d}x$. Since the rate of heat generation is uniform along the length of the train L, we have that the rate of heat given to the track element is $mav\frac{\mathrm{d}x}{L}$, where m is the train's mass, a is the train's deceleration, and v is the trains speed. Integrating over time gives the total heat given to the track element: $\mathrm{d}Q = ma\frac{\mathrm{d}x}{L}\Delta x$, where Δx is the total distance the train slips on the track element. Combining with $\mathrm{d}Q = c\mathrm{d}x \cdot \Delta T$, we get $T(x) = T_0 + \frac{ma}{cL}\Delta x$, where c is the linear heat capacity. Now we split into 3 cases:

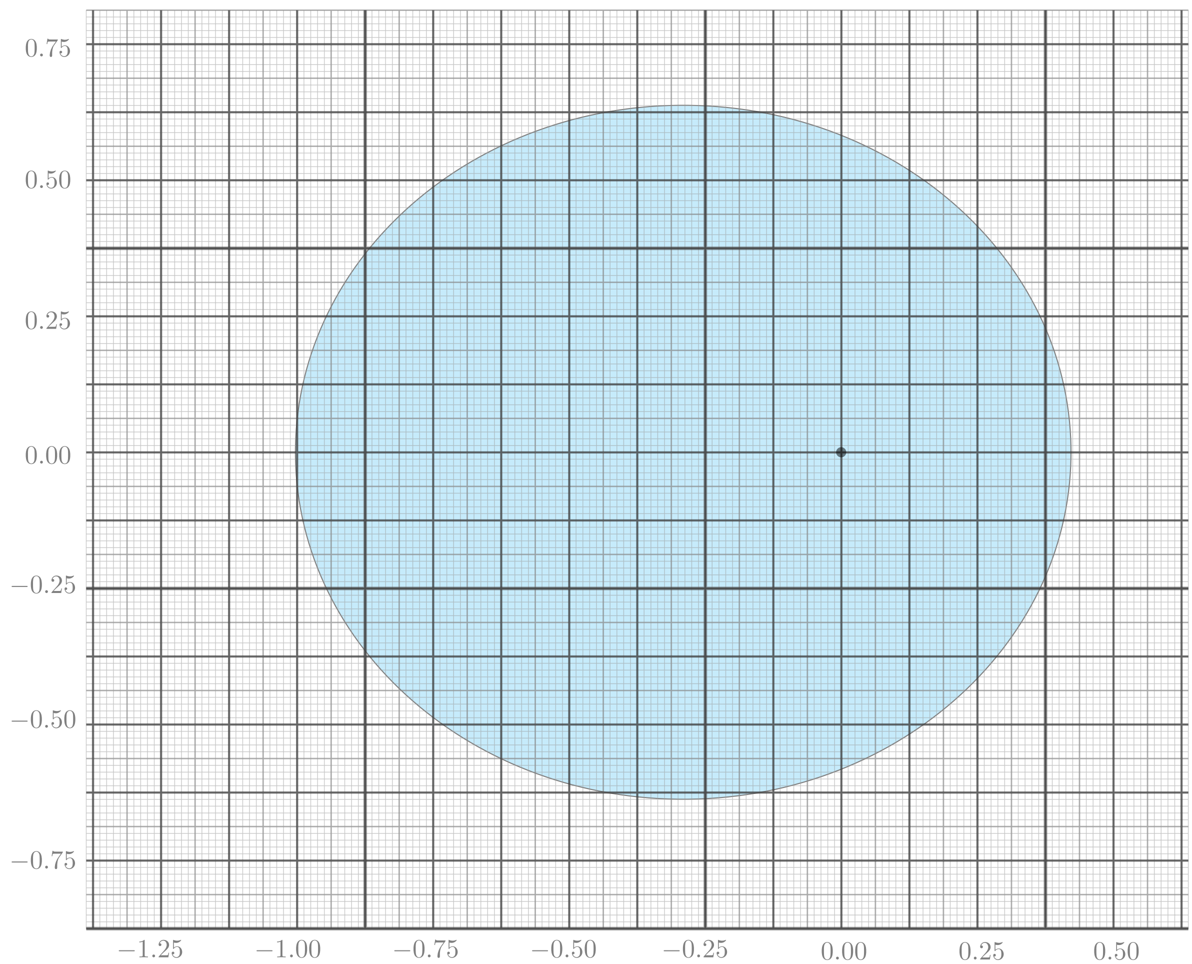
- $0 < x < L$,
- $L < x < d$,
- $d < x < d + L$,

where d is the total distance. If $0 < x < L$, the train slips a distance $\Delta x = x$ on the track element, so we have $T(x) = T_0 + \frac{ma}{cL}x$ (linear). If $L < x < d$, the train slips a distance $\Delta x = L$ on the track element, so we have $T(x) = T_0 + \frac{ma}{c}$ (constant). If $d < x < d + L$, the train slips a distance $\Delta x = d + L - x$, so we have $T(x) = T0 + \frac{ma}{cL}(d + L - x)$ (linear). Note that $a = \frac{v_0^2}{2d}$ where v_0 is the train's initial velocity.

$\boxed{63.98^\circ \text{ C}}$

8. Fountain A sprinkler fountain is in the shape of a semi-sphere that spews out water from all angles at a uniform speed v such that without the presence of wind, the wetted region around the fountain forms a circle in the XY plane with the fountain centered on it.

Now suppose there is a constant wind blowing in a direction parallel to the ground such that the force acting on each water molecule is proportional to their weight. The wetted region forms the shape below where the fountain is placed at $(0, 0)$. Determine the exit speed of water v in meters per second. Round to two significant digits. All dimensions are in meters.



Solution:
Method One: Let us look at only the major axis x. The wind provides a constant acceleration along x, which makes this problem equivalent to throwing a ball up and down a ramp tilted at an angle of $\beta = \tan \frac{a}{g}$. The furthest point along the x axis represents the maximum distance you can throw an object up and down this ramp. This distance can be derived to be:

$$d_{\text{optimal}} = \frac{v^2/g}{1 + \sin \beta} \tag{6}$$

where β is negative if it's tilting downwards. We get the systems of equations:

$$\frac{v^2/g_{\text{eff}}}{1+\sin\beta} = 0.25 + 0.25 \cdot \frac{13}{20} \tag{7}$$

$$\frac{v^2/g_{\text{eff}}}{1-\sin\beta} = 1 \tag{8}$$

where:

$$g_{\text{eff}} = g/\cos\beta \tag{9}$$

and solving gives $v = 2.51\,\text{m}\,\text{s}^{-1}$.
Method Two: Looking at the minor axis is much easier. The wind doesn't contribute in this direction, so we know the furthest point must have a vertical displacement of $d_{\text{max}} = \frac{v^2}{g}$. We measure $d_{\text{max}} = 0.50 + 0.25 \cdot \frac{11}{20}$ and solving for v gives $d_{\text{max}} = 2.50$ m.
Note that the answers aren't exactly the same due to measurement inaccuracies, so we asked for two significant digits. An earlier version of the question had the same diagram, but the labels were accidentally shifted. This did not affect the final answer.

$\boxed{2.5 \text{ m/s}}$

9. Escaping Nieons Consider a gas of mysterious particles called nieons that all travel at the same speed, v. They are enclosed in a cubical box, and there are ρ nieons per unit volume. A very small hole of area A is punched in the side of the box. The number of nieons that escape the box per unit time is given by

$$\alpha v^\beta A^\gamma \rho^\delta \tag{10}$$

where α, β, γ, and δ are all dimensionless constants. Calculate $\alpha + \beta + \gamma + \delta$.

Solution: The main idea is this: if a nieon is to escape in time Δt, then it must be traveling towards the hole and be within a hemisphere of radius $v\Delta t$, centered on the hole.

(Note that we will ignore all collisions between nieons. As with many calculations in these kinds of problems, collisions will not affect our answer. Even if they did, we can assume that the hole is so small that we can take $v\Delta t \to 0$, making the radius of the hemisphere much much smaller than the mean free path.)

First, we will break up this hemisphere into volume elements. We will calculate how many nieons are inside each volume element. Then we will find out how many of these nieons are traveling in the direction of the hole.

We define a spherical coordinate system (r, θ, ϕ). The volume of each differential element is $r^2 \sin\theta dr d\theta d\phi$, which means that there are $\rho r^2 \sin\theta dr d\theta d\phi$ nieons inside this volume element.

Within this volume element, how many of these nieons are going towards the hole? We must think about what each nieon "sees." Visualize a sphere of radius r around the volume element. The surface of the sphere will pass through the small hole in the wall. Each nieon is equally likely to go in any direction and is equally likely to end up going through any patch of this spherical surface. Thus, each nieon within this volume element has a probability $A'/4\pi r^2$ of going through the hole, where A' is the *perceived* area of the hole from where the nieon is situated.

A' can be calculated as follows. $\hat{\mathbf{j}}$ is the unit vector pointing perpendicular to the hole. Let $\hat{\mathbf{r}}$ be the unit vector pointing from the hole to the volume element. Then it can be verified that $A' = A\cos\psi$, where ψ is the angle between $\hat{\mathbf{j}}$ and $\hat{\mathbf{r}}$.

Expressing $\hat{\mathbf{r}}$ in terms of the other unit vectors, it can be shown that $A' = A\sin\theta\sin\phi$. Hence, the fraction of nieons inside the volume element that are going towards the hole is

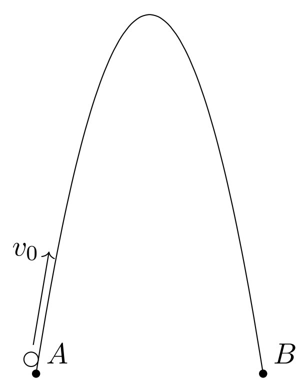
$$\frac{A\sin\theta\sin\phi}{4\pi r^2}.$$

Integrating over the entire hemisphere, the number of nieons escaping through the hole in a time Δt is

$$\int_0^{v\Delta t} dr \int_0^{\pi} d\phi \int_0^{\pi} d\theta \left(\frac{A}{4\pi r^2}\sin\theta\sin\phi\right)(\rho r^2\sin\theta) = \frac{1}{4}vA\rho\cdot\Delta t.$$

3.25

10. Pico-Pico 1 Poncho is a very good player of the legendary carnival game known as Pico-Pico. Its setup consists of a steel ball, represented by a point mass, of negligible radius and a frictionless vertical track. The goal of Pico-Pico is to flick the ball from the beginning of the track (point A) such that it is able to traverse through the track while never leaving the track, successfully reaching the end (point B). The most famous track design is one of parabolic shape; specifically, the giant track is of the shape $h(x) = 5 - 2x^2$ in meters. The starting and ending points of the tracks are where the two points where the track intersects $y = 0$. If $(v_a, v_b]$ is the range of the ball's initial velocity v_0 that satisfies the winning condition of Pico-Pico, help Poncho find $v_b - v_a$. This part is depicted below:



Solution: Using conservation of energy, the minimum initial velocity of the ball needed to pass the top of the track is $v_a = \sqrt{2gh} = 9.9045\frac{m}{s}$. To find v_b, the centripetal force at all points on the track must be determined given the initial velocity.

$$F_c = \frac{mv^2}{R} \tag{11}$$

$$= \frac{m(v_b^2 - 2gh)}{\frac{|1+(\frac{d}{dx}h(x))^2|}{\frac{d^2}{dx^2}h(x)}} \tag{12}$$

$$= \frac{m(v_b^2 - 2gh)}{\frac{|1+16x^2|^{\frac{3}{2}}}{4}} \tag{13}$$

For the boundary condition, the ball leaves if the normal force from the track on the ball $N =$

$mg\cos\theta - F_c$ becomes 0.

$$mg\cos\theta - F_c = 0 \tag{14}$$

$$mg\cos\arctan(-4x) = \frac{4m(v_b^2 - 2gh)}{|1+16x^2|^{\frac{3}{2}}} \tag{15}$$

$$\frac{g}{|1+16x^2|^{\frac{1}{2}}} = \frac{4(v_b^2 - 2gh)}{|1+16x^2|^{\frac{3}{2}}} \tag{16}$$

$$g = \frac{4(v_b^2 - 2gh)}{1+16x^2} \tag{17}$$

$$v_{b\mathbf{max}} = \sqrt{\frac{g+16gx^2}{4} + 2gh} \tag{18}$$

From the derivation, $v_{b\mathbf{max}}$ is the lowest at $x = 0$. Thus,

$$v_{b\mathbf{max}} = \sqrt{\frac{g}{4} + 2gh} \tag{19}$$

$$= 10.0276\frac{m}{s} \tag{20}$$

which is our desired v_b. The final answer, $v_b - v_a$, can be calculated.

0.1231

11. Pico-Pico 2 Now, Poncho has encountered a different Pico-Pico game that uses the same shaped frictionless track, but lays it horizontally on a table with friction and coefficient of friction $\mu = 0.8$. In addition, the ball, which can once again be considered a point mass, is placed on the other side of the track as the ball in part 1. Finally, a buzzer on the other side of the track requires the mass to hit with at least velocity $v_f = 2$ m/s in order to trigger the buzzer and win the game. Find the minimum velocity v_0 required for the ball to reach the end of the track with a velocity of at least v_f. The initial velocity must be directed along the track.

Solution: We simply need to find the work done by friction and we can finish with conservation of energy. To get the work done by friction, since the force is constant, we just need to find the arc length of the track. That is

$$\ell = \int_{-\sqrt{5/2}}^{\sqrt{5/2}} \sqrt{1 + h'(x)^2}\,\mathrm{d}x = \int_{-\sqrt{5/2}}^{\sqrt{5/2}} \sqrt{1+16x^2}\,\mathrm{d}x = 10.76 \text{ m}.$$

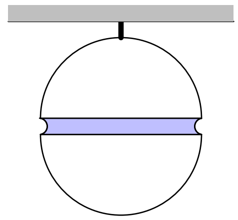
Then by conservation of energy,

$$\frac{1}{2}v_0^2 - \mu g\ell = \frac{1}{2}v_f^2 \implies v_0 = \sqrt{v_f^2 + 2\mu g\ell}.$$

We simply plug in the numbers to finish.

13.1 m/s

12. Golden Apple Anyone who's had an apple may know that pieces of an apple stick together, when picking up one piece a second piece may also come with the first piece. The same idea is tried on a *golden apple*. Consider two uniform hemispheres with radius $r = 4$ cm made of gold of density $\rho_g = 19300\text{kg}\,\text{m}^{-3}$. The top half is nailed to a support and the space between is filled with water.



Given that the surface tension of water is $\gamma = 0.072\text{N}\,\text{m}^{-1}$ and that the contact angle between gold and water is $\theta = 10°$, what is the maximum distance between the two hemispheres so that the bottom half doesn't fall? Answer in millimeters.

Solution: Let h be the difference in height. There are 3 forces on the bottom hemisphere. The force from gravity, which has magnitude $\frac{2}{3}\pi\rho_g g r^3$, the force from the surface tension, and the force from the pressure difference at the top and bottom. The pressure difference is given by the young-laplace equation,

$$\Delta P = \gamma\left(-\frac{1}{r} + \frac{2\cos\theta}{h}\right) \approx \frac{2\gamma\cos\theta}{h}.$$

The radii are found by some simple geometry. It is likely that r will be much larger than the height, so we can neglect the $1/r$ term. Now the force from surface tension is $2\pi r\gamma\sin\theta$, since we take the vertical component. So we can now set the net force to 0,

$$\frac{2}{3}\pi\rho_g g r^3 = \pi r^2\Delta P + 2\pi r\gamma\sin\theta \implies \frac{2}{3}\rho_g g r^2 = (2r\gamma\cos\theta)\frac{1}{h} + 2\gamma\sin\theta.$$

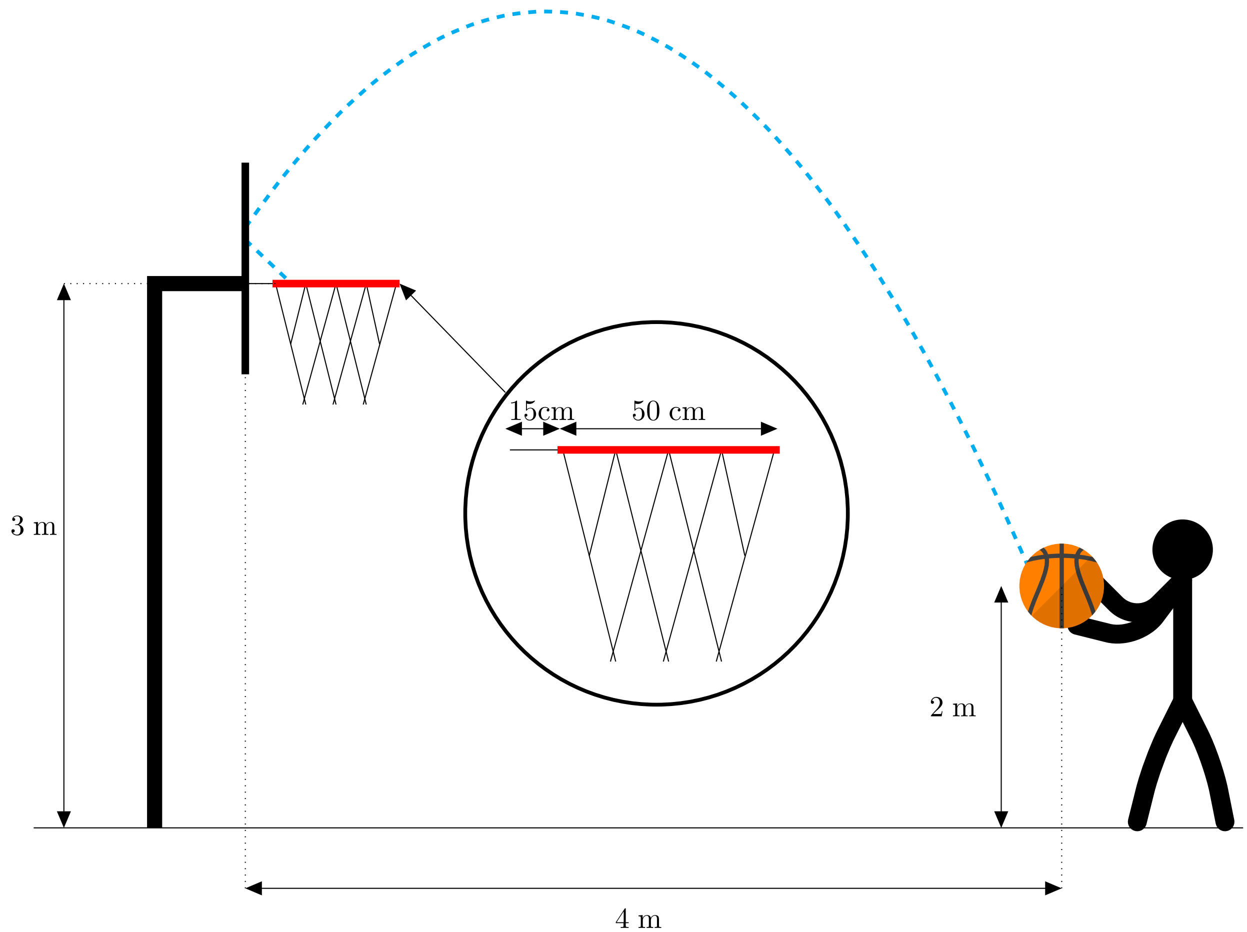
This is simple to solve,

$$h = \frac{2r\gamma\cos\theta}{2\rho_g g r^2/3 - 2\gamma\sin\theta} = 2.81\times10^{-5}\text{ m} = 0.0281\text{ mm}.$$

This is very small so our approximation from earlier is justified.

0.0281 mm

The following information applies to the next 2 problems. In the following two problems we will look at shooting a basketball. Model the basketball as an elastic hollow sphere with radius 0.1 meters. Model the net and basket as shown below, dimensions marked. Neglect friction between the backboard and basketball, and assume all collisions are perfectly elastic.



13. Free Throw For this problem, you launch the basketball from the point that is 2 meters above the ground and 4 meters from the backboard as shown. You attempt to make a shot by hitting the basketball off the backboard as depicted above. What is the minimum initial speed required for the ball to make this shot?

Note: For this problem, you may assume that the size of the ball is negligible.

Solution: For this part, we regard the basketball as a point mass. Now to account for the backboard bounce, we reflect the hoop over the backboard. The problem is now equivalent to making it into the hoop that is 4.15 meters away now. So now we need to find the smallest velocity to reach this point. One way to do it is to use the safety parabola, which makes it so minimal calculations are needed, so we present that method here. However, this problem is also readily solvable with only basic kinematics.

The point that the ball must hit (4.15 meters in front and 1 meter above) lies on the safety parabola when it is thrown at minimum speed. Since the focus of the safety parabola is the point of launch, the directrix is

$$\sqrt{1^2 + 4.15^2} = 4.27 \text{ m}$$

away from the top of the hoop. The distance from the focus to the vertex of the parabola is then $(4.27 + 1)/2 = 2.63$ m. So the minimum velocity to go this height is given by

$$\frac{v^2}{2g} = 2.63\,\text{m},$$

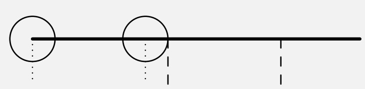
and that gives $v = 7.19\,\text{m/s}$. There is one last bit we need to check to complete this problem. The basketball's trajectory has to actually be possible (i.e. it doesn't go underneath the rim). It is clearly less optimal intuitively. If it passes through the underside, it means that its apex of the trajectory is very near its target.

$\boxed{7.19 \text{ m/s}}$

14. Layup You now wish to practice closer shots. You walk up until you're 1 m away from the backboard (the 4 m changes to a 1 m). You jump 1 m in the air. What is the minimum initial speed of the ball that allows you to score off of the backboard if you release the ball at the top of your jump? Note that scoring off the backboard means that the ball bounces off the backboard and into the net. Do not consider cases where the ball bounces off of the rim or the protrusion. That's just luck and you want a consistent strategy.

Hint: Neglecting the size of the ball may no longer be possible.

Solution: For this part, first notice that the shot is now from a much closer location. This means we must take into account the radius of the sphere, which we'll call r. We first reflect over the entire setup over a vertical line r away from the backboard. The reason it doesn't reflect over the backboard is because the center of the ball bounces before that. Now the basketball must also clear the rim since it's not a point mass. So we can draw a circle of radius r centered at the end of the rim. So the problem is equivalent to clearing the circles shown. Note that we cannot simply set the range equal to 1.05 m because the ball would intersect the rim earlier when it's at an angle (many teams sent challenges with solutions assuming that incorrect fact).



Now consider the minimum case where it clears it. Then the parabola is going to be tangent to the closer circle. Furthermore, the safety parabola will also be tangent because at the boundaries the trajectory parabolas are tangent to the safety parabola. This is the key insight. Now we use geometry.

Consider the center of the right circle. Then suppose the parabola is tangent at some angle θ on the circle from the vertical counterclockwise. Then, the line from the focus of the safety parabola must also make an angle θ with the parabola because that's how a parabola reflects lines from the directrix and focus. Thus, with a little trig, we see that

$$(r\cos\theta)\tan 2\theta = d + r\sin\theta$$

where d is the distance from the launch point to the center of the circle. This distance is in fact just $1-0.05=0.95$ meters. Now we jsut do a bit of algebra to finish:

$$r\frac{2\sin\theta}{1-\tan^2\theta}=d+r\sin\theta.$$

Rearranging,

$$r\sin\theta\frac{1+\tan^2\theta}{1-\tan^2\theta}=d.$$

Notice that $1+\tan^2\theta=\sec^2\theta$, and so

$$r\frac{\sin\theta}{\cos^2\theta-\sin^2\theta}=d.$$

This is a quadratic in sin,

$$2\sin^2\theta+\frac{r}{d}\sin\theta-1=0.$$

Using the values for r and d and the quadratic formula, we arrive at $\sin\theta\approx 0.681$. Thus, $\theta=42.9°$. At this point we can now find the distance from the focus to the vertex. The distance from the focus to the point of tangency is

$$\sqrt{(d+r\sin\theta)^2+(r\cos\theta)^2}\approx 1.02\,\mathrm{m}.$$

So the distance to the directrix is that plus $r\sin\theta$, so the directrix is at $1.09\,\mathrm{m}$ above the axis containing the top of the baskets and the launch point. So the velocity is the same as the velocity required to go up 1.09/2 meters from the launch point, so we set

$$\frac{v^2}{2g}=1.09/2\implies v_{min}=3.27\text{ m/s}.$$

There are some things we need to check though. We need to make sure the ball doesn't curve back in quickly enough to hit the rim, but this is pretty unlikely to occur considering the curvatures, the circle has a much smaller radius of curvature than the parabola there, so it won't go back and hit it. A quick drawing also confirms that the safety parabola doesn't hit the other rim, so we can see that the trajectory won't hit the other rim either.

$\boxed{3.27\text{ m/s}}$

15. Right Triangle Potentials Let ABC be a solid right triangle ($AB=5s$, $AC=12s$, and $BC=13s$) with uniform charge density σ. Let D be the midpoint of BC. We denote the electric potential of a point P by $\phi(P)$. The electric potential at infinity is 0. If $\phi(B)+\phi(C)+\phi(D)=\frac{k\sigma s}{\epsilon_0}$ where k is a dimensionless constant, determine k.

Solution: If we put two of these right triangles together, we can form a rectangle with side lengths $5s$ and $12s$. Let V be the potential at the center of this rectangle. By superposition, $\phi(D)=\frac{V}{2}$. Consider the potential at the corner. It can be decomposed into the potential from each of the right triangles, which is precisely $\phi(B)+\phi(C)$. Now, also note that if we put 4 of these rectangles side by side, to make a larger rectangle with side lengths $10s$ and $24s$, the potential is scaled by a factor of 2 due to dimensional analysis arguments. Thus, the potential at the center of this larger rectangle is $2V$, but this can be decomposed into the sum of the potentials at the

corners of the 4 smaller rectangles. Thus, the potential in the corner of the smaller rectangle is $\frac{V}{2} = \phi(B) + \phi(C)$. Thus, we obtain $\phi(B) + \phi(C) + \phi(D) = V$.
Now, we will find the potential at the center of the rectangle. Note that it suffices to find the potential at the vertex of an isosceles triangle because we can connect the center of the rectangle to the 4 corners to create 4 isosceles triangles. Suppose we have an isosceles triangle with base $2x$ and height y. The potential at the vertex is

$$\int\int \frac{\sigma}{4\pi\epsilon_0 r}(rdrd\theta) = \frac{\sigma}{4\pi\epsilon_0}\int_{-\tan^{-1}(\frac{x}{y})}^{\tan^{-1}(\frac{x}{y})}\int_0^{\frac{y}{\cos\theta}} drd\theta = \frac{\sigma y}{2\pi\epsilon_0}\log\left(\frac{x+\sqrt{x^2+y^2}}{y}\right).$$

If the sides of the rectangle are a and b, we then obtain the potential at the center is

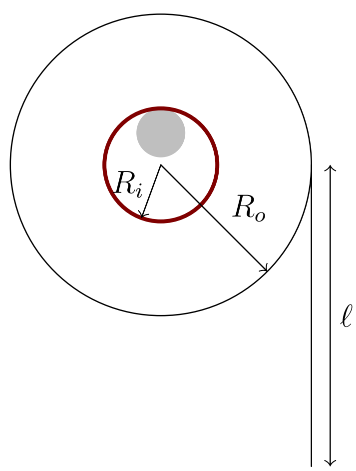
$$V = \frac{\sigma}{2\pi\epsilon_0}\left(a\log\left(\frac{b+\sqrt{a^2+b^2}}{a}\right) + b\log\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)\right).$$

In this case,

$$k = \frac{5}{2\pi}\ln 5 + \frac{6}{\pi}\ln\frac{3}{2}.$$

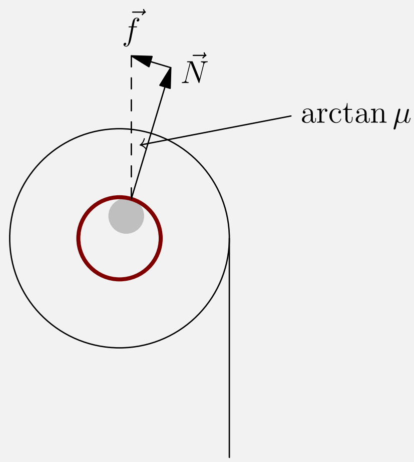
2.055

16. Toilet Paper Roll Consider a toilet paper roll with some length of it hanging off as shown. The toilet paper roll rests on a cylindrical pole of radius $r = 1$ cm and the coefficient of static friction between the role and the pole is $\mu = 0.3$.



The length of the paper hanging off has length $\ell = 30$ cm and the inner radius of the roll is $R_i = 2$ cm. The toilet paper has thickness $s = 0.1$ mm and mass per unit length $\lambda = 5$ g/m. What is the minimum outer radius R_o such that the toilet paper roll remains static? Answer in centimeters.

Solution: Due to the length of toilet paper hanging off, the toilet paper will be slightly tilted, in order for torques to balance. The tilt isn't shown in the diagram, since it is meant to be found. So the tilted normal force has to be compensated by the frictional force, and just before slippage, it will look something like:



Let $\theta = \arctan\mu$. So now balancing torques about the contact point, if m is the mass of the toilet paper roll,

$$mgR_i \sin\theta = (\ell\lambda)g(R_o - R_i \sin\theta).$$

And now we need m. The mass per unit area is $\sigma = \lambda/s$, so $m = \pi(R_o^2 - R_i^2)\lambda/s$. So substituting this in,

$$\frac{\pi(R_o^2 - R_i^2)\lambda}{s} gR_i \sin\theta = \ell\lambda g(R_o - R_i \sin\theta).$$

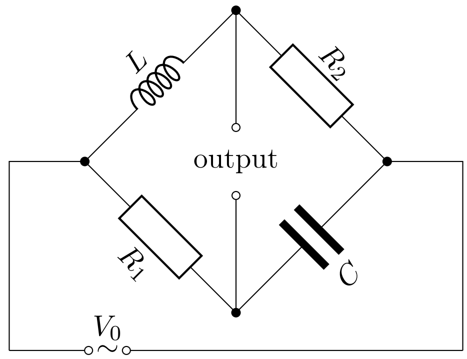
Simplifying a bit,

$$\pi(R_o^2 - R_i^2)R_i \sin\theta = s\ell\left(R_o - R_i \sin\theta\right).$$

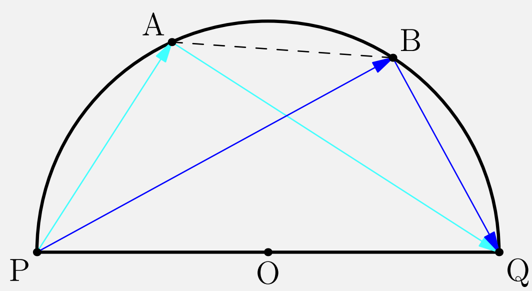
This is a quadratic and we can just plug in the numbers and use the quadratic formula or use a graphing calculator to finish, yielding $R_o = 2.061$ cm.

$\boxed{2.061 \text{ cm}}$

17. Maximum Voltage In the circuit shown below, a capacitor $C = 4\text{F}$, inductor $L = 5\text{H}$, and resistors $R_1 = 3\Omega$ and $R_2 = 2\Omega$ are placed in a diamond shape and are then fed an alternating current with peak voltage $V_0 = 1\text{V}$ of unknown frequency. Determine the magnitude of the maximum instantaneous output voltage shown in the diagram.



Solution: We use the method of phasors. Consider the following phasor diagram:



We define angles as $\angle APB = \angle AQB = \gamma$, $\angle AOB = 2\gamma$, $\angle BPQ = \alpha$, $\angle AQP = \beta$. Note that $\overrightarrow{PA} = I_2\omega L$, $\overrightarrow{AQ} = I_2 R$, $\overrightarrow{PB} = I_1 R_1$, $\overrightarrow{BQ} = 1/I_1\omega C$. We seek to maximize the length of AB. We can write via law of cosines that

$$\text{AB} = \sqrt{r^2 + r^2 - 2r\cos(2\gamma)} = 2r\sin(\gamma).$$

where $r = V_0$ is the radius of the circle of which phasors are inscribed in. Since $\gamma = \alpha + \beta$, we can then rewrite the length of AB to be

$$\text{AB} = r(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = V_0\left(\frac{R_1R_2}{V_0^2} - \frac{L}{CV_0^2}\right)I_1I_2 = V_0\left(R_1R_2 - \frac{L}{C}\right)\left(\frac{1}{|Z_1||Z_2|}\right)$$

where the product of both complex exponentials is simply

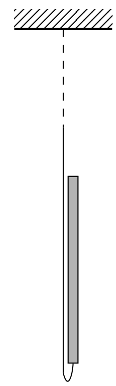
$$|Z_1||Z_2| = \sqrt{\omega^2L^2 + R_2^2}\sqrt{\frac{1}{\omega^2C^2} + R_1^2}$$

which implies the best frequency of the circuit is $\omega = \sqrt{\frac{R_2^2}{L^2R_1C}}$. This means with algebra, the lowest value of $|Z_1||Z_2| = \frac{L}{C} - R_1R_2$. Hence, the maximum value of AB is simply

$$\text{AB} = V_0\left(\frac{R_1R_2 - \frac{L}{C}}{R_1R_2 + \frac{L}{C}}\right).$$

0.65 V

18. Suspended Rod - 1 A uniform bar of length l and mass m is connected to a very long thread of negligible mass suspended from a ceiling. It is then rotated such that it is vertically upside down and then released. Initially, the rod is in unstable equilibrium. As it falls down, the minimum tension acting on the thread over the rod's entire motion is given by αmg. Determine α.



Solution: First note that the thread is given to be very long. Therefore, only vertical tension and gravitational forces act on the rod allowing for its center of mass to move in a straight vertical line. Using this fact, we can now write the acceleration of the rod in terms of angular velocity ω and angular acceleration ε. Let the angle of the rod at any moment to the vertical be φ. For simplicity, we define the length of the rod to be $2l$. Then, by defining the coordinate y to be the change in vertical length of the rod where $y = l\cos\varphi$, one can write for varying $\varphi \in [\varphi, \varphi + \mathrm{d}\varphi)$ that $l\cos(\varphi + \mathrm{d}\varphi) = y + \mathrm{d}y \implies \mathrm{d}y = l\sin\varphi \mathrm{d}\varphi$. Thus, simple differentiation proves that

$$v = l\sin\varphi\omega \implies a = l\varepsilon\sin\varphi + l\omega^2\cos\varphi.$$

We can now also write conservation of energy to get another relationship between velocity and angular velocity. At any given moment, it can be written that $mgl = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - mgl\cos\varphi$. Since $I = \frac{1}{2}m(2l)^2 = \frac{1}{3}ml^2$, then

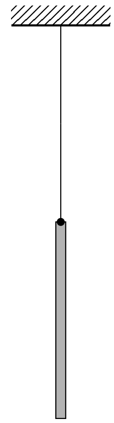
$$mgl(1+\cos\varphi) = ml^2\omega^2\left(\frac{1}{2}\sin^2\varphi + \frac{1}{6}\right).$$

Our third equation comes from the fact that the tension on the rod at any moment can be written as $T = mg - ma \implies ma = mg - T$. We can finally get a fourth equation by equating torques such that $\frac{1}{3}ml^2\varepsilon = Tl\sin\varphi$. With these four equations, simple algebra yields the tension at any point is

$$T(\varphi) = \frac{1 + (3\cos\varphi - 1)^2}{(1+3\sin^2\varphi)^2}mg \implies T_{\min} \approx 0.165mg.$$

0.165

19. Suspended Rod - 2 A uniform bar of length l and mass m is connected to a thread of length $2l$ of negligible mass and is suspended from the ceiling at equilibrium. The rod is then slightly nudged at a point on its body. The largest stable frequency of oscillations of the system is given by $\beta\sqrt{\frac{g}{l}}$. Determine β.



Solution: Let the angle the rod makes with the vertical be θ, and let the angle the string makes with the vertical be ϕ. Note that the angles are in opposite directions.Throughout the motion, the tension in the string remains at approximately mg. Furthermore, we assume the motion is in simple harmonic oscillation, so $\ddot{\theta} = -\omega^2\theta$ and $\ddot{\phi} = -\omega^2\phi$. Writing the force and torque equations of the rod gives

$$mg\phi = m\left(2l\omega^2\phi - \frac{l}{2}\omega^2\theta\right)$$
$$mg(\theta+\phi)\frac{l}{2} = \frac{1}{12}ml^2\omega^2\theta$$

Solving this gives us a quartic equation $l^2\omega^4 - 8gl\omega^2 + 3g^2 = 0$ which implies that $\omega^2 = (4 \pm \sqrt{13})\frac{g}{l}$. Thus, the largest mode of frequency is $f_{\max} = \frac{1}{2\pi}\sqrt{4+\sqrt{13}\frac{g}{l}}$. However, note that a nudge could be interpreted as giving the rod a slight torque or a slight impulse. Which, in the later case would imply, with some calculations, that the largest oscillation frequency is not physically significant as the impulse would be imparted farther away than the rods length. Thus, we accepted either root for this problem.

$$\boxed{\frac{1}{2\pi}\sqrt{4 \pm \sqrt{13}}}$$

20. One Ladder A straight ladder AB of mass $m = 1$ kg is positioned almost vertically such that point B is in contact with the ground with a coefficient of friction $\mu = 0.15$. It is given an infinitesimal kick at the point A so that the ladder begins rotating about point B. Find the value ϕ_m of angle ϕ of the ladder with the vertical at which the lower end B starts slipping on the ground.

Solution: By conservation of energy, we have $\frac{1}{2}I\omega_m^2 = mg\frac{L}{2}(1-\cos\phi_m)$ where $I = \frac{1}{3}mL^2$. Thus,

$$\omega_m = \sqrt{\frac{3g(1-\cos\phi_m)}{L}}.$$

Also, by torque analysis about B, we have $\tau = mg\frac{L}{2}\sin\phi_m = I\alpha_m$ which means

$$\alpha_m = \frac{3g}{2L}\sin\phi_m.$$

Thus, the centripetal and tangential accelerations of the ladder are $a_c = \omega_m^2\frac{L}{2} = \frac{3}{2}g(1-\cos\phi_m)$ and $a_t = \alpha_m\frac{L}{2} = \frac{3}{4}g\sin\phi_m$ respectively. The normal force is thus $N = mg - ma_c\cos\phi - ma_t\sin\phi$, so

$$\frac{N}{mg} = 1 - \frac{3}{2}\cos\phi_m(1-\cos\phi_m) - \frac{3}{4}\sin^2\phi_m.$$

The frictional force is thus $f = ma_t\cos\phi_m - ma_c\sin\phi_m$ so

$$\frac{f}{mg} = \frac{3}{4}\sin\phi_m\cos\phi_m - \frac{3}{2}\sin\phi_m(1-\cos\phi_m).$$

Setting $\frac{f}{N} = \mu$, we have $6\sin\phi_m(1-\cos\phi_m) - 3\sin\phi_m\cos\phi_m = -4\mu + 6\mu\cos\phi_m(1-\cos\phi_m) + 3\mu\sin^2\phi_m$. Simplifying

$$6\sin\phi_m - 9\sin\phi_m\cos\phi_m + 9\mu\cos^2\phi_m - 6\mu\cos\phi_m = -\mu.$$

We then can solve for ϕ_m numerically.

$\boxed{11.5^\circ}$

21. Two Ladders Two straight ladders AB and CD, each with length 1 m, are symmetrically placed on smooth ground, leaning on each other, such that they are touching with their ends B and C, ends A and D are touching the floor. The friction at any two surfaces is negligible. Initially both ladders are almost parallel and vertical. Find the distance AD when the points B and C lose contact.

Solution: The center of mass of both of the ladders moves in a circle, centered at the point on the ground directly beneath B/C. So we find when the required normal force between the two ladders is 0. That is, when the total net force on one of the ladders is when the two ladders lose contact. Let $2r = \ell$. Now by conservation of energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{3}\frac{v^2}{r^2} = mgr(1-\cos\theta),$$

where θ is defined as the angle the ladder makes witht he vertical. So we have

$$v^2\left(1+\frac{1}{3}\right) = 2gr(1-\cos\theta) \implies v^2 = \frac{3}{2}gr(1-\cos\theta).$$

So the centripetal acceleration is

$$a_c = \frac{v^2}{r} = \frac{3}{2}g(1-\cos\theta).$$

And the tangential acceleration is

$$\frac{dv}{dt} = \sqrt{\frac{3gr}{2}} \frac{\sin\theta}{2\sqrt{1-\cos\theta}} \frac{d\theta}{dt}.$$

And $\frac{d\theta}{dt} = \frac{v}{r}$, so

$$a_\theta = \frac{dv}{dt} = \frac{3g}{2}\frac{\sin\theta}{2}.$$

Now for the total acceleration to be vertical, we need

$$a_c \tan\theta = a_\theta,$$

so

$$1 - \cos\theta = \frac{\cos\theta}{2}.$$

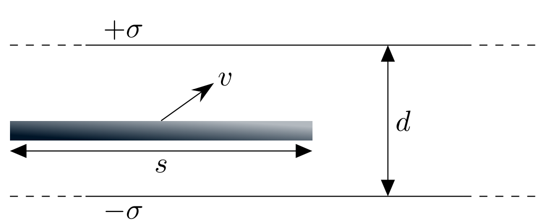
This simplifies to

$$2 = 3\cos\theta.$$

So the distance between the two ends is $4r\sin\theta = 4r\sqrt{1-\frac{4}{9}} = \frac{4r\sqrt{5}}{3}$.

$\boxed{1.491 \text{ m}}$

22. Colliding Conducting Slab A thin conducting square slab with side length $s = 5\,\text{cm}$, initial charge $q = 0.1\,\mu\text{C}$, and mass $m = 100\,\text{g}$ is given a kick and sent bouncing between two infinite conducting plates separated by a distance $d = 0.5\,\text{cm} \ll s$ and with surface charge density $\pm\sigma = \pm 50\,\mu\text{C/m}^2$. After a long time it is observed exactly in the middle of the two plates to be traveling with velocity of magnitude $v = 3\,\text{m/s}$ and direction $\theta = 30^\circ$ with respect to the horizontal line parallel to the plates. How many collisions occur after it has traveled a distance $L = 15\,\text{m}$ horizontally from when it was last observed? Assume that all collisions are elastic, and neglect induced charges. Note that the setup is horizontal so gravity does not need to be accounted for.



Solution: After the first collision, the charge approaches a constant magnitude. Let's look at a collision with the plate at charge q hitting the $-\sigma$ plate. Since the slab is thin, to keep the conducting surface an equipotential, the charge on the conduting slab has to become $-\sigma s^2$. The initial charge doesn't matter because the plates are infinite. So the charge is $-\sigma s^2$ after leaving the negative plate and $+\sigma s^2$ after leaving the positive plate.
Now the acceleration always has a constant magnitude, which is directed toward the plate it is traveling to, which we can find using $F = ma$,

$$(\sigma s^2)\frac{\sigma}{\epsilon_0} = ma \implies a = \frac{\sigma^2 s^2}{\epsilon_0 m} = 7.06\,\text{m/s}^2.$$

Then here we can use the trick of reflecting over the plates, and now it's a basic kinematics problem. The time to travel L is $t = \frac{L}{v\cos\theta}$, and then the distance it travels vertically is

$$y = v\sin\theta\frac{L}{v\cos\theta} + \frac{1}{2}a\left(\frac{L}{v\cos\theta}\right)^2 = 126.36\,\mathrm{m}.$$

And since $25272d + d/2 \approx 87.129$, the number of collisions is 25273.

$\boxed{25273}$

23. Evil Gamma Photon An evil gamma photon of energy $E_{\gamma 1} = 200$ keV is heading towards a spaceship. The commander's only choice is shooting another photon in the direction of the gamma photon such that they 'collide' head on and produce an electron-positron pair (both have mass m_e). Find the lower bound on the energy $E_{\gamma 2}$ of the photon as imposed by the principles of special relativity such that this occurs. Answer in keV.

Solution: The key claim is that energy is minimized when both particles are moving at the same velocity after the collision. This can be proved by transforming into the frame where the total momentum is 0.
This idea is sufficient because it implies both the electron and positron have the same momentum and energy after the collision. Let them both have momentum p. We then have

$$2p = \frac{E_{\gamma_1} - E_{\gamma_2}}{c} \implies pc = \frac{E_{\gamma_1} - E_{\gamma_2}}{2}. \tag{21}$$

By energy conservation, both the electron and positron have energy $\frac{E_{\gamma_1}+E_{\gamma_2}}{2}$. Using the result $E = (pc)^2 + (mc^2)^2$, we obtain,

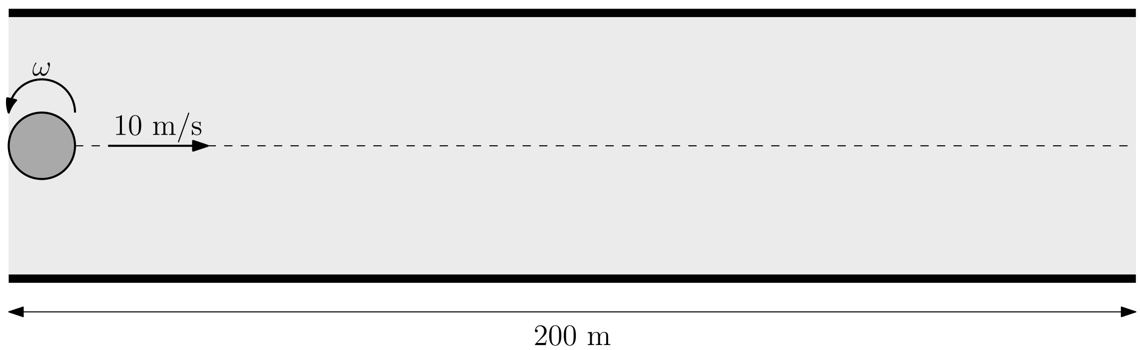
$$\left(\frac{E_{\gamma_1} + E_{\gamma_2}}{2}\right)^2 = (pc)^2 + (m_e c^2)^2. \tag{22}$$

Combining equation 21 and equation 22, we get

$$\left(\frac{E_{\gamma_1} + E_{\gamma_2}}{2}\right)^2 = \left(\frac{E_{\gamma_1} - E_{\gamma_2}}{2}\right)^2 + (m_e c^2)^2 \implies E_{\gamma_2} = \frac{m_e^2 c^4}{E_{\gamma_1}}.$$

$\boxed{1306 \text{ keV}}$

24. Spinning Cylinder Adithya has a solid cylinder of mass $M = 10\,\text{kg}$, radius $R = 0.08\,\text{m}$, and height $H = 0.20\,\text{m}$. He is running a test in a chamber on Earth over a distance of $d = 200\,\text{m}$ as shown below. Assume that the physical length of the chamber is much greater than d (i.e. the chamber extends far to the left and right of the testing area). The chamber is filled with an ideal fluid with uniform density $\rho = 700\,\text{kg/m}^3$. Adithya's cylinder is launched with linear velocity $v = 10\,\text{m/s}$ and spins counterclockwise with angular velocity ω. Adithya notices that the cylinder continues on a **horizontal path** until the end of the chamber. Find the angular velocity ω. Do not neglect forces due to fluid pressure differences. Note that the diagram presents a side view of the chamber (i.e. gravity is oriented downwards with respect to the diagram).



Assume the following about the setup and the ideal fluid:

- fluid flow is steady in the frame of the center of mass of the cylinder
- the ideal fluid is incompressible, irrotational, and has zero viscosity
- the angular velocity of the cylinder is approximately constant during its subsequent motion

Hint: For a uniform **cylinder** of radius R rotating counterclockwise at angular velocity ω situated in an ideal fluid with flow velocity u to the **right** far away from the cylinder, the velocity potential Φ is given by

$$\Phi(r, \theta) = ur\cos\theta + u\frac{R^2}{r}\cos\theta + \frac{\Gamma\theta}{2\pi}$$

where (r, θ) is the polar coordinate system with origin at the center of the cylinder. Γ is the circulation and is equal to $2\pi R^2\omega$. The fluid velocity is given by

$$\mathbf{v} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}.$$

Solution: We will work in the reference frame of the center of mass of the cylinder because the fluid flow is steady in this reference frame. The key intuition here is that the magnitude of the fluid velocity above the cylinder will be higher on the top because the tangential velocity of the cylinder is in the same direction as the velocity of the fluid on the top. By Bernoulli's principle, this means that the pressure on the top is lower than the pressure on the bottom, which will create a lift force on the cylinder.
With the given theory, we can model this quantitatively. In our chosen reference frame, the water

moves with velocity v to the left. The velocity potential around a cylinder with radius R is

$$\Phi(r,\theta) = -vr\cos\theta - v\frac{R^2}{r}\cos\theta + R^2\omega\theta.$$

Therefore, we find

$$\mathbf{v} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} = -v\left(1 - \frac{R^2}{r^2}\right)\cos\theta\hat{\mathbf{r}} + \left(v\left(1 + \frac{R^2}{r^2}\right)\sin\theta + R\omega\right)\hat{\theta}.$$

As expected from boundary conditions, the radial velocity vanishes when $r = R$. Furthermore, on the surface of the cylinder, we have the tangential velocity of the fluid is $2v\sin\theta + R\omega$ in the counterclockwise direction. Consider points on the cylinder at angles θ and $-\theta$. By Bernoulli's principle (ignoring the height difference which will be accounted with the buoyant force),

$$p_{-\theta} - p_\theta = \frac{1}{2}\rho((2v\sin\theta + R\omega)^2 - (-2v\sin\theta + R\omega)^2) = 4\rho vR\omega\sin\theta.$$

If we integrate this result along the surface of the cylinder, we can find the lift force per unit length. Note that only the vertical components of the pressure will matter as the horizontal components cancel due to symmetry. The vertical component of the pressure difference is then $4\rho v r_0\omega\sin^2\theta$. Thus, the lift force per unit length is

$$\frac{F_{\text{lift}}}{H} = \int_0^\pi 4\rho vR\omega\sin^2(\theta)(Rd\theta) = 2\pi\rho\omega vR^2.$$

The total left force is

$$F_{\text{lift}} = 2\pi\rho R^2 H\omega v.$$

The gravitational force is Mg, and the buoyant force is $\pi R^2 H\rho g$. Therefore, we must have

$$\pi R^2 H\rho g + 2\pi\rho R^2 H\omega v = Mg.$$

Solving for ω, we obtain

$$\omega = \frac{Mg}{2\pi R^2 H\rho v} - \frac{g}{2v}.$$

$\boxed{1.25\,\mathrm{s}^{-1}}$

25. Optimal Launch Adithya is launching a package from New York City (40°43′ N and 73°56′ W) to Guam (13°27′ N and 144°48′ E). Find the minimal launch velocity v_0 from New York City to Guam. Ignore the rotation of the earth, effects due to the atmosphere, and the gravitational force from the sun. Additionally, assume the Earth is a perfect sphere with radius $R_\oplus = 6.37 \times 10^6$ m and mass $M_\oplus = 5.97 \times 10^{24}$ kg.

Solution: We first want to find the angular distance between New York City and Guam. Let this be θ. Let New York City be point A and Guam be point B. Consider the north pole P and the spherical triangle PAB. By the spherical law of cosines,

$$\cos\theta = \cos(90^\circ - \phi_A)\cos(90^\circ - \phi_B) + \sin(90^\circ - \phi_A)\sin(90^\circ - \phi_B)\cos(\ell_B - \ell_A). \tag{23}$$

Equation 23 simplifies to

$$\cos\theta = \sin\phi_A \sin\phi_B + \cos\phi_A \cos\phi_B \cos(\ell_B - \ell_A) \tag{24}$$

from which we find $\theta = 115.05°$.
Now that we have determined the angular distance, we will proceed with the orbital mechanics problem. By the vis-viva equation, the speed at the launch point is

$$v_0 = \sqrt{GM_\oplus \left(\frac{2}{R_\oplus} - \frac{1}{a}\right)} \tag{25}$$

where a is the semimajor axis for the orbit. It is clear that in order to minimize v_0, we must minimize a. The orbit is an ellipse with focii F_1 and F_2, where F_1 is the center of the earth. By the definition of an ellipse,

$$AF_1 + AF_2 = 2a. \tag{26}$$

Since $AF_1 = R_\oplus$, is suffices to minimize AF_2. By symmetry, line F_1F_2 is the perpendicular bisector of AB. Since F_2 is on a fixed line, to minimize F_2, we place it at the foot of the perpendicular from A to this line. Then, we obtain $AF_2 = R_\oplus \sin\left(\frac{\theta}{2}\right)$. Thus,

$$a = \frac{1 + \sin\left(\frac{\theta}{2}\right)}{2} R_\oplus,$$

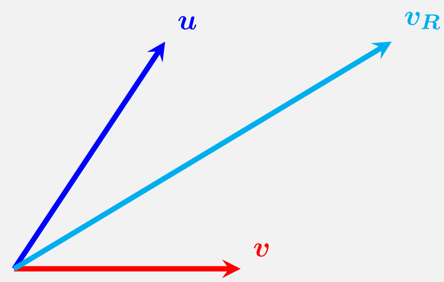
and from Equation 25 we find

$$v_0 = \sqrt{\frac{GM_\oplus}{R_\oplus}\frac{2\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}}.$$

$\boxed{7564\,\mathrm{m/s}}$

26. Drag on the Plate Consider a container filled with argon, with molar mass $39.9\mathrm{g\,mol^{-1}}$ whose pressure is much smaller than that of atmospheric pressure. Suppose there is a plate of area $A = 10\mathrm{mm}^2$ moving with a speed v perpendicular to its plane. If the gas has density $\rho = 4.8 \times 10^{-7}\mathrm{g\,cm^{-3}}$, and temperature $T = 100\mathrm{K}$, find an approximate value for the drag force acting on the plate. Suppose that the speed of the plate is $v = 100\mathrm{m\,s^{-1}}$.

Solution: Let N be the number of particles colliding with the plate in a unit time, and let all the molecules arrive at the plate with the same speed u. The force on the plate will be the momentum imparted over a unit time, i.e. $F = 2Nm(v-u)/t$ where the coefficient off two reflects on the fact that the average speed of the molecules hitting the wall is the same as the molecules departing. Note that Maxwells distribution dictates that the average speed of particles in all direction $\langle u\rangle = 0$ which means that the average force acting on the plate is simply $\langle F\rangle = 2Nmv/t$. During a time period t, these molecules arrive a wall that is of thickness $v_R t$ where v_R is the relative velocity between the molecules and the plate (which is not negligible as the mean free path and velocity of the plate are comparable.) This means the number of collisions found in this layer will be $N = \frac{1}{2}nV \approx Av_R t$ where the factor of 1/2 reflects on how half the molecules go to the plate and the other half go the other direction. Thus the force acting on the plate will become $\langle F_D\rangle = 2mvN/t = 2nmv \cdot v_R A$. To find the relative velocity, consider the vector diagram:



By law of cosines, the magnitude of v_R will simply be

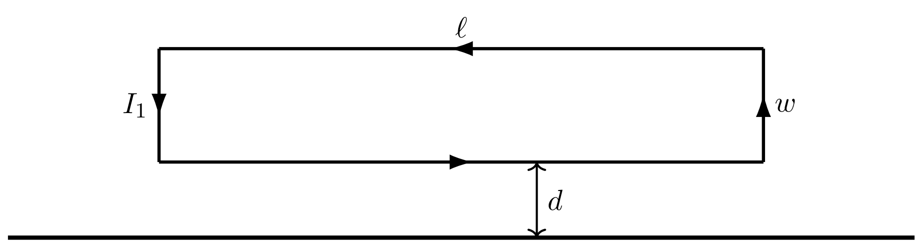
$$v_R^2 = v^2 + u^2 - 2\boldsymbol{v} \cdot \boldsymbol{u}.$$

The direction of $\boldsymbol{u}$ points homogeneously in all directions as the orientation of molecules changes with each individual one. Therefore, $2\boldsymbol{v} \cdot \boldsymbol{u}$ will point in all directions averaging to 0. Thus, the magnitude of v_R will simply be $\langle v_R \rangle = \sqrt{v^2 + \langle u \rangle^2}$ where $\langle u \rangle^2$ is the average thermal velocity of molecules. In terms of density, we can then express the drag force to be $2\rho A v \sqrt{v^2 + \langle u \rangle^2}$.

$\boxed{2.41 \times 10^{-4} \text{ N}}$.

27. Superconducting Loop Consider a rectangular loop made of superconducting material with length $\ell = 200$ cm and width $w = 2$ cm. The radius of this particular wire is $r = 0.5$ mm. This superconducting rectangular loop initially has a current $I_1 = 5$A in the counterclockwise direction as shown in the figure below. This rectangular loop is situated a distance $d = 1$ cm above an infinitely long wire that initially contains no current. Suppose that the current in the infinitely long wire is increased to some current I_2 such that there is an attractive force F between the rectangular loop and the long wire. Find the maximum possible value of F. Write your answer in newtons.

Hint: You may neglect the magnetic field produced by the vertical segments in the rectangular loop.



Solution: The key idea is that the superconducting loop must have constant flux. If it did not, by Faraday's Law, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

would be generated in the loop. Since superconducting materials have no resistance, this would imply an infinite current, hence a contradiction.

We will first compute the flux through the rectangular loop when there is a current I_1. Since $w \ll \ell$, we can assume that the vertical segments produce negligible amounts of magnetic field. We can furthermore approximate the field produced by one of the horizontal wires a distance r away as

$\frac{\mu_0 I}{2\pi r}$ (this is valid for an infinitely long wire, and therefore is also valid in the regime where $w \ll \ell$). Thus, the total flux through the rectangular loop when there is a current I_1 is

$$\Phi_1 = \int_r^w B(\ell dr') = \int_r^{w-r} \left(\frac{\mu_0 I_1}{2\pi r'} + \frac{\mu_0 I_1}{2\pi(w-r')} \right) \ell dr' = \frac{\mu_0 I_1 \ell}{\pi} \ln\left(\frac{w}{r}\right).$$

Note that the self inductance of the loop is $L = \frac{\Phi}{I_1} = \frac{\mu_0 \ell}{\pi} \ln\left(\frac{w}{r}\right)$.
Now, we will determine the flux through the rectangular loop due to the long current-carrying wire. This is

$$\Phi_2 = \int_d^{d+w} \frac{\mu_0 I_2}{2\pi r}(\ell dr) = \frac{\mu_0 I_2 \ell}{2\pi} \ln\left(\frac{d+w}{d}\right).$$

The mutual inductance is $M = \frac{\Phi_2}{I_2} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{d+w}{d}\right)$. In to maintain the same flux in the loop, the current will change to I_3 where

$$LI_1 = MI_2 + LI_3,$$

or

$$I_3 = I_1 - \frac{M}{L} I_2.$$

Now, we compute the force between the rectangular loop and the long, current-carrying wire. The forces on the vertical sides cancel out because the current in the loop is in opposite directions on these sides. From the horizontal sides, we have the force is

$$F = \sum (I_3 \vec{\ell} \times \vec{B}) = I_3 \ell \left(\frac{\mu_0 I_2}{2\pi d} - \frac{\mu_0 I_2}{2\pi(d+w)} \right)$$
$$= \frac{\mu_0 \ell w}{2\pi d(d+w)} \left[I_2 \left(I_1 - \frac{M}{L} I_2 \right) \right].$$

This quadratic in I_2 is maximized when $I_2 = \frac{L}{2M} I_1$ in which case the force becomes

$$F = \frac{\mu_0 \ell w}{2\pi d(d+w)} \frac{L I_1^2}{4M} = \frac{\mu_0 \ell w I_1^2}{4\pi d(d+w)} \frac{\ln\left(\frac{w}{r}\right)}{\ln\left(\frac{d+w}{d}\right)}.$$

$\boxed{1.12 \times 10^{-3} \text{ N}}$

Note: If the size of the wires is considered when computing flux, a slightly different answer is obtained. In the contest, all answers between 1.11×10^{-3} and 1.18×10^{-3} were accepted.

28. Cantor Interference Consider a 1 cm long slit with negligible height. First, we divide the slit into thirds and cover the middle third. Then, we perform the same steps on the two shorter slits. Again, we perform the same steps on the four even shorter slits and continue for a very long time.

Then, we shine a monochromatic, coherent light source of wavelength 500 nm on our slits, which creates an interference pattern on a wall 10 meters away. On the wall, what is the distance between the central maximum and the first side maximum? Assume the distance to the wall is much greater than the width of the slit. Answer in millimeters.

Solution: This problem is essentially an infinite convolution. In detail, consider two separate amplitude functions $f(x)$ and $g(x)$ that correspond to two different interference patterns. One can think of a convolution of f and g as taking f and placing the same function in every position that exists in g. To be more specific, the amplitude of one interference pattern can be thought of as the product of the amplitude patterns of two *separate* amplitude functions. For example, the diffraction pattern of two slits is the same as the product of the amplitudes for one slit and two light sources (this will be given as an exercise to prove). To make more sense of this, let us consider that very example and let f pertain to the amplitude function of a single slit and let g pertain to the amplitude function of two light sources. One can then write in phase space with a generalized coordinate x' that $\{f * g\}(x) = \int_{-\infty}^{\infty} f(x - x')g(x')\mathrm{d}x'$ according to the convolution theorem. The integral then runs through all values and places a copy of $f(x - x')$ at the peaks of $g(x')$.
With this in hand, we can use the convolution theorem to our advantage. Let us designate the amplitude function of the entire Cantor slit to be $F(\theta)$. Since θ is small, we can designate the phase angle as $\phi = kx\theta$ where x is a variable moving through all of the slits. If we consider a single slit that is cut into a third, the angle produced will be a third of its original as well, or the new amplitude function will simply be a function of $\theta/3$ or $F(\theta/3)$. The distance between the midpoints of the two slits will simply be $d = 2/3$ cm which then becomes $2/9$, then $2/27$ and so on. In terms of the original function, we can decompose it into the function of $F(\theta/3)$ and the amplitude function of a single light source. In other words, $F(\theta) = F(\theta/3)\cos(kd\theta/2)$ where $k = \frac{2\pi}{\lambda}$. To the limit of infinity, this can be otherwise written as

$$F(\theta) = \prod_{i=0}^{\infty} \cos\left(\frac{kd\theta}{2 \cdot 3^i}\right) = \cos\left(\frac{kd\theta}{2}\right)\cos\left(\frac{kd\theta}{6}\right)\cos\left(\frac{kd\theta}{18}\right)\dots$$

Finding an exact mathematical solution would be difficult, thus it is simple enough to just multiply the first 4 or 5 terms to achieve the final answer.

$\boxed{0.647 \text{ mm}}$

The following information applies to the next 2 problems. A certain planet with radius $R = 3 \times 10^4$ km is made of a liquid with constant density $\rho = 1.5$ g/cm^3 with the exception of a homogeneous solid core of radius $r = 10$ km and mass $m = 2.4 \times 10^{16}$ kg. Normally, the core is situated at the geometric center of the planet. However, a small disturbance has moved the center of the core $x = 1$ km away from the geometric center of the planet. The core is released from rest, and the fluid is inviscid and incompressible.

29. Solid Core - 1 Calculate the magnitude of the force due to gravity that now acts on the core. Work under the assumption that $R \gg r$.

Solution: We solve by simplifying the configuration into progressively simpler but mathematically equivalent formulations of the same problem.

Because the core is spherical and homogeneous, all the gravitational forces on the core are equivalent to the gravitational forces on a point mass located at the center of the core. Thus, the problem is now the force of gravity on a point mass of mass m located x distance away from the center of planet, with a sphere of radius r evacuated around the point mass.

But if we filled this evacuated sphere, would the force on the point mass be any different? No! If we filled up the sphere, all the added liquid (of the same density as the rest of the planet) would add no additional force on the point mass because it is symmetrically distributed around the point mass. Thus, the problem is now the force of gravity on a point mass of mass m located x distance away from the center of the planet.

By shell theorem, we can ignore all the fluid that is more than a distance x away from the center of the planet. Thus, the problem is now the force of gravity on a point mass m situated on the surface of a liquid planet of radius x. This force is not difficult to calculate at all:

$$\begin{aligned} F &= \frac{GMm}{x^2} \\ &= \frac{Gm}{x^2}\left(\frac{4}{3}\pi x^3\rho\right) \\ &= \frac{4}{3}\pi Gm\rho x \end{aligned}$$

$\boxed{1.0058 \times 10^{13}\ \mathrm{N}}$

30. Solid Core - 2 Calculate the magnitude of the force due to the pressure from the liquid that now acts on the core.

Solution:
Let ρ_1 be the density of the planet and ρ_2 be the density of the core. Let the center of the planet be A, and the center of the core be B. Let the gravitational potential energy of system be $U(x)$. Initially, the core and fluid are stationary, but due to the gravitational and pressure forces on the core, the core will accelerate towards A, with some acceleration a. Note that since the mass of the planet is much greater than that of the core, the planet's CM and shape will be affected negligibly. After a small time t, conservation of energy gives $\frac{1}{2}\mathcal{M}(at)^2 = \Delta U = U'(x) \cdot \frac{1}{2}at^2$, where $\mathcal{M}$ takes into account the mass of the core and also the motion of the fluid around the core. Once we determine $\mathcal{M}$ and $U(x)$, we can solve for the acceleration of the core, which will give the total force on the core. Then we subtract off the gravitational force calculated in Part 1 to get the pressure force.

The effective mass $\mathcal{M}$ is given by $\rho_2 \cdot \frac{4}{3}\pi r^3 + \rho_1 \cdot \frac{2}{3}\pi r^3$, where the first term is the actual mass and the second term is the added mass, see here. Thus, $\mathcal{M} = (2\rho_2 + \rho_1) \cdot \frac{2}{3}\pi r^3$.
To find the gravitational potential energy $U(x)$, we consider how much work it takes to move the particles infinitely away. We do the following procedure:

- Step 1: Compress a sphere of density $\rho_2 - \rho_1$ and radius r centered at B so that it becomes nearly a point mass at B.
- Step 2: Move the "point mass" infinitely far away from the planet.
- Step 3: Expand the "point mass" until it becomes a sphere of radius r again.
- Step 4: Expand both the planet and the sphere infinitely, OR simply calculate the gravitational potential energy of this simple arrangement.

Before we do any steps yet, first note that to solve the problem we only need to analyze Steps 1 and 2 because the other steps do not involve x and can be treated as constants (since we are interested in $U'(x)$). Here we include the other steps for completeness.

We first tackle Step 1. Note that the gravitational field inside a uniform solid sphere of density ρ is $\mathbf{g} = -\frac{4\pi G}{3}\rho\mathbf{r}$, where $\mathbf{r}$ is the position vector from the center. Let $\mathbf{v}_1 = \overrightarrow{\mathrm{AB}}$ and $\mathbf{v}_2 = \mathbf{r} - \mathbf{v}_1$, so $\mathbf{g} = -\frac{4\pi G}{3}\rho_1(\mathbf{v}_1 + \mathbf{v}_2)$. Suppose Agent 1 is responsible for countering the force due to $\mathbf{v}_1$, and Agent 2 for $\mathbf{v}_2$. Say Agent 3 is responsible for countering the force due to the sphere's own self gravity. Then, Agent 1 does no work, because the CM of the sphere does not change throughout Step 1. The work Agent 2 does W_2 can be computed by noting that the potential associated with the force due to $\mathbf{v}_2$ is $\frac{2\pi G}{3}\rho r^2$. Thus,

$$W_2 = -\int_0^r (\rho_2 - \rho_1) \cdot 4\pi r^2 \cdot \frac{2\pi G}{3}\rho_1 r^2 dr = -\frac{8\pi^2 G}{15}\rho_1(\rho_2 - \rho_1)r^5.$$

For now, we defined the work done by Agent 3 to be W_3. The total work done in Step 1 is thus $W_2 + W_3 = -\frac{8\pi^2 G}{15}\rho_1(\rho_2 - \rho_1)r^5 + W_3$

Now, we do Step 2. The work it takes to move the point mass to the edge of the planet is $\frac{2\pi G}{3}\rho_1(R^2 - x^2) \cdot \frac{4}{3}\pi r^3(\rho_2 - \rho_1)$. The work it takes to further move the point mass to infinity is $G \cdot \rho_1 \cdot \frac{4}{3}\pi R^2 \cdot \frac{4}{3}\pi r^3(\rho_2 - \rho_1)$. Thus, the total work in Step 2 is $\frac{8\pi^2 G}{9}\rho_1(\rho_2 - \rho_1)r^3(3R^2 - x^2)$.

Next, we do Step 3. The work it takes to expand the point mass back to a sphere of radius r is simply $-W_3$.

Finally, we do Step 4. The potential energy of a uniform sphere of density ρ and radius r is $-\frac{16}{15}\pi^2 G\rho^2 R^5$. Thus, the total potential energy of the planet and the sphere is $-\frac{16}{15}\pi^2 G\rho_1^2 R^5 - \frac{16}{15}\pi^2 G(\rho_2 - \rho_1)^2 r^5$.

Combining all steps gives $U(x) = \frac{8\pi^2 G}{9}\rho_1(\rho_2 - \rho_1)r^3 x^2 + C$, where C is some constant independent of x. We have

$$a = \frac{U'(x)}{\mathcal{M}} = \frac{16\pi^2 G\rho_1(\rho_2 - \rho_1)r^3 x}{9 \cdot (2\rho_2 + \rho_1) \cdot \frac{2}{3}\pi r^3} = \frac{8\pi^2 G\rho_1(\rho_2 - \rho_1)x}{3(2\rho_2 + \rho_1)}.$$

The gravitational force from part 1 was $F_g = \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x$. Thus, the pressure force is

$$\begin{aligned} F_p = F_g - Ma &= \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x - \frac{4}{3}\pi r^3\rho_2 \cdot \frac{8\pi^2 G\rho_1(\rho_2 - \rho_1)x}{3(2\rho_2 + \rho_1)} \\ &= \frac{16}{9}\pi^2 G\rho_1\rho_2 r^3 x(1 - 2 \cdot \frac{\rho_2 - \rho_1}{2\rho_2 + \rho_1}) = \frac{16}{3}\pi^2 G r^3 x \cdot \frac{\rho_1^2\rho_2}{2\rho_2 + \rho_1}. \end{aligned}$$

$\boxed{3.4926 \times 10^{12}\ \mathrm{N}}$

31. Solenoids A scientist is doing an experiment with a setup consisting of 2 ideal solenoids that share the same axis. The lengths of the solenoids are both ℓ, the radii of the solenoids are r and $2r$, and the smaller solenoid is completely inside the larger one. Suppose that the solenoids share the same (constant) current I, but the inner solenoid has $4N$ loops while the outer one has N, and they have opposite polarities (meaning the current is clockwise in one solenoid but counterclockwise in the other).

Model the Earth's magnetic field as one produced by a magnetic dipole centered in the Earth's core.

Let F be the magnitude of the total magnetic force the whole setup feels due to Earth's magnetic field. Now the scientist replaces the setup with a similar one: the only differences are that the the radii of the solenoids are $2r$ (inner) and $3r$ (outer), the length of each solenoid is 7ℓ, and the number of loops each solenoid is $27N$ (inner) and $12N$ (outer). The scientist now drives a constant current $2I$ through the setup (the solenoids still have opposite polarities), and the whole setup feels a total force of magnitude F' due to the Earth's magnetic field. Assuming the new setup was in the same location on Earth and had the same orientation as the old one, find F'/F.

Assume the dimensions of the solenoids are much smaller than the radius of the Earth.

Solution: We can solve the problem by assuming that the location of the setup is at the North Pole and that the solenoids are oriented so that their axis intersects the Earth's core. Note that if we had some other location or orientation, then both F and F' would be multiplied by the same factor, so their ratio remains the same.

Suppose the radii of the solenoids are r and αr, where the number of inner and outer loops are N and $\frac{N}{\alpha^2}$, respectively. To find the force the Earth's dipole exerts on the solenoids, we can calculate the force the solenoids exert on the dipole. To do this, we need to find the gradient of the magnetic field produced by the solenoids at the dipole's location. Let the radius of the Earth be R.

Consider the field produced by 2 concentric, coaxial, ring currents, the inner ring with current I radius r and the outer one with current $\frac{I}{\alpha^2}$ and radius αr. The currents are in opposite directions. At a distance R away from the center of the rings, along their axis, the magnetic field is given by

$$\begin{aligned} B &= \frac{\mu_0 I r^2}{2(R^2+r^2)^{\frac{3}{2}}} - \frac{\mu_0 I r^2}{2(R^2+(\alpha r)^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 I r^2}{2R^3}\left(\left(1+\frac{r^2}{R^2}\right)^{-\frac{3}{2}} - \left(1+\frac{\alpha^2 r^2}{R^2}\right)^{-\frac{3}{2}}\right) \\ &\approx \frac{\mu_0 I r^2}{2R^3}\left(\frac{3}{2}(\alpha^2-1)\frac{r^2}{R^2}\right) \\ &= \frac{3\mu_0 I r^4}{4R^5}(\alpha^2-1) \end{aligned}$$

Thus, the gradient of the magnetic field is proportional to $Ir^4(\alpha^2-1)$. Now we consider the actual setup. The new setup multiplies the effective current $\frac{27}{4}\cdot\frac{2}{1} = \frac{27}{2}$ times, while multiplying r by 2. The factor α^2-1 changed from 3 to $\frac{5}{4}$. Combining, we get $\frac{F'}{F} = \frac{27}{2}\cdot 2^4\cdot\frac{5}{12} = 90$.

$\boxed{90}$

The following information applies to the next 2 problems. Adithya is in a rocket with proper acceleration $a_0 = 3.00\times 10^8\ \mathrm{m/s^2}$ to the right, and Eddie is in a rocket with proper acceleration $\frac{a_0}{2}$ to the left. Let the frame of Adithya's rocket be S_1, and the frame of Eddie's rocket be S_2. Initially, both rockets are at rest with respect to each other, they are at the same location, and Adithya's clock and Eddie's clock are both set to 0.

32. Accelerating Rockets - 1 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **velocity** of Adithya's rocket in S_2?

Solution: Throughout this solution, we will let $c = a_0 = 1$, for simplicity. We will work in the inertial frame that is initially at rest with both rockets at $t_1 = t_2 = 0$. First, we determine the velocity of Adithya's rocket in this frame as a function of the proper time that has elapsed t_1. In frame. S_1, in a time dt_1, the rocket's velocity increases by dt_1, so by velocity addition, the new velocity in the inertial frame is

$$\frac{v_1 + dt_1}{1 + v_1 dt_1} \approx v_1 + (1 - v_1^2)dt_1.$$

Therefore, we have

$$\frac{dv_1}{dt_1} = 1 - v_1^2.$$

Upon separating and integrating, we find $v_1 = \tanh(t_1)$. Similarly, the velocity of Eddie's rocket in the inertial frame is $v_2 = \tanh(t_2/2)$. Now, in this inertial frame, let the time between events A and B be t, and let the distance between rockets A and B be x. By a Lorentz transformation, the time between the events in frame S_2 is

$$t' = \gamma(t - v_2 x) = 0$$

since the events are simultaneous in S_2. Therefore, we must have $t = v_2 x$. Note that $t = t_A - t_B$ where t_A and t_B denote the times of events A and B in the inertial frame, respectively. Also, note that $x = x_A + x_B$ where x_A and x_B are the respective displacements of the two rockets in the inertial frame. By the effects of time dilation, we have

$$t_A = \int \gamma dt_1 = \int \frac{1}{\sqrt{1 - v_1^2}} dt_1 = \int_0^{t_1} \cosh\left(t_1'\right) dt_1' = \sinh(t_1).$$

Similarly, $t_B = 2\sinh(t_2/2)$, and we obtain $t = 2\sinh(t_2/2) - \sinh(t_1)$. Additionally, from the above result,

$$x_A = \int v_1 dt = \int v_1 \cosh(t_1) dt_1 = \int_0^{t_1} \sinh(t_1) dt_1' = \cosh(t_1) - 1.$$

Similarly, $x_B = 2\cosh(t_2/2) - 2$, and $x = \cosh(t_1) + 2\cosh(t_2/2) - 3$. Thus, from $t = v_2 x$,

$$2\sinh(t_2/2) - \sinh(t_1) = \tanh(t_2/2)(\cosh(t_1) + 2\cosh(t_2/2) - 3).$$

$$-\sinh(t_1) = \tanh(t_2/2)(\cosh(t_1) - 3).$$

$$\tanh(t_2/2) = \frac{\sinh(t_1)}{3 - \cosh(t_1)}.$$

Now, this is the velocity of Eddie's rocket as measured from the inertial frame, so by velocity addition, the velocity of Adithya's rocket as seen by Eddie is

$$v = \frac{\tanh(t_1) + \frac{\sinh(t_1)}{3 - \cosh(t_1)}}{1 + \frac{\sinh(t_1)\tanh(t_1)}{3 - \cosh(t_1)}} = \frac{3\tanh(t_1)}{3 - \cosh(t_1) + \sinh(t_1)\tanh(t_1)}.$$

$\boxed{0.855377c = 2.564 \times 10^8 \text{ m/s}}$.

33. Accelerating Rockets - 2 At the moment Adithya's clock reaches 0.75 s in S_2, what is the **acceleration** of Adithya's rocket in S_2?

Solution: Observe that Eddie measures the following acceleration:

$$a = \frac{dv}{dt_2} = \frac{dv}{dt_1} \cdot \frac{dt_1}{dt_2}.$$

Both of these results can be determined from the equations above. Note that from quotient rule:

$$\frac{dv}{dt_1} = \frac{3\,\mathrm{sech}^2(t_1)(3 - \cosh(t_1))}{(3 - \cosh(t_1) + \sinh(t_1)\tanh(t_1))^2} = 0.6151.$$

Additionally, we obtain

$$\frac{dt_1}{dt_2} = \frac{\mathrm{sech}^2(t_2/2)(3 - \cosh(t_1))^2}{2(3\cosh(t_1) - 1)} = 0.3869.$$

$\boxed{7.14 \cdot 10^7 \text{ m/s}^2}$.

The following information applies to the next 2 problems. Suppose a ping pong ball of radius R, thickness t, made out of a material with density ρ_b, and Young's modulus Y, is hit so that it resonates in mid-air with small amplitude oscillations. Assume $t \ll R$. The density of air around (and inside) the ball is ρ_a, and the air pressure is p, where $\rho_a \ll \rho_b \frac{t}{R}$ and $p \ll Y \frac{t^3}{R^3}$.

34. Ping Pong - 1 An estimate for the resonance frequency is $\omega \sim R^a t^b \rho_b^c Y^d$. Find the value of $4a^2 + 3b^2 + 2c^2 + d^2$.

Hint: The surface of the ball will oscillate by "bending" instead of "stretching", since the former takes much less energy than the latter.

Solution: Throughout the problem, we will work to order of magnitude and ignore prefactors.

The hint says the surface of the ball will bend instead of stretch, so we need to develop a theory of bending. First, we consider the simplified scenario of a long beam with thickness t, width w, length L, made out of a material with Young's modulus Y and density ρ. When the beam bends, the top part of the beam will be in tension and the bottom part will be in compression. Thus, this is how the potential energy is stored in the beam. Suppose the beam is bent with curvature κ. Then the top part of the beam will stretch by $Lt\kappa$, and the bottom part will compress by the same amount. Using Hooke's law, we can approximate the total potential energy stored in the beam as $U \sim \frac{Ytw}{L}(Lt\kappa)^2 \sim Yt^3wL\kappa^2$. Note that if the relaxed state of the beam was already curved, we simply replace κ with the change in curvature.

To find the oscillation frequency of the beam, we need to find the kinetic energy in terms of $\dot{\kappa}$. Since curvature is on the order of second derivative of displacement, we can multiply κ by L^2 to get an estimate for displacement. Then $\dot{\kappa}L^2$ gives an estimate for speed, so the kinetic energy is $K \sim \rho twL(\dot{\kappa}L^2)^2 \sim \rho twL^5\dot{\kappa}^2$. Thus, the frequency of oscillations is $\omega \sim \sqrt{\frac{Yt^2}{\rho L^4}}$. Again, if the beam was already curved, we can replace κ everywhere with the change in curvature.

We can model the ping pong ball as a "beam" of length order R, width order R, and thickness t. This is a very crude approximation, but will give a dimensionally correct answer (since we are ignoring prefactors). The angular frequency is thus $\omega \sim \frac{t}{R^2}\sqrt{\frac{Y}{\rho_b}}$.

$\boxed{19.75}$

35. Ping Pong - 2 Assuming that the ball loses mechanical energy only through the surrounding air, find an estimate of the characteristic time τ it takes for the ball to stop resonating (or to lose half its mechanical energy), that is $\tau \sim R^\alpha t^\beta \rho_b^\kappa Y^\delta \rho_a^\zeta p^\gamma$. Find the value of $6\alpha^2 + 5\beta^2 + 4\kappa^2 + 3\delta^2 + 2\zeta^2 + \gamma^2$. (Note that in reality, the ball also loses mechanical energy to heat, but we will neglect that for simplicity.)

Solution:
The ping pong ball loses energy through the surrounding sound waves. Let the surface of the ball oscillate with amplitude on the order of A. Then the energy stored in the ball is on the order of kinetic energy, which is on the order of $E \sim \rho_b R^2 t A^2 \omega^2$. The sound waves have intensity $I \sim \rho_a \omega^2 A^2 c_s$, where the speed of sound $c_s \sim \sqrt{\frac{p}{\rho_a}}$. Thus, the total energy lost per unit time is $P \sim IR^2 \sim \sqrt{\rho_a p} A^2 \omega^2 R^2$. Finally, the characteristic time $\tau \sim \frac{E}{P} \sim \frac{\rho_b t}{\sqrt{\rho_a p}}$.

$\boxed{9.75}$