2021 Online Physics Olympiad: Invitational Contest

v1.1 Theoretical Exam

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Instructions for Theoretical Exam

The theoretical examination consists of 4 long answer questions and 100 points over 2 full days from August 13, 0:01 am GMT.

- The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Each question in this examination are equally worth 25 points. Be sure to spend your time wisely.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in $\LaTeX$. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade $\LaTeX$ template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
Universal gas constant	$R = 8.31$ J/(mol $\cdot$ K)
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	1 eV $= 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}$ (N $\cdot$ m^2)/kg^2
Acceleration due to gravity	$g = 9.81$ m/s^2
1 unified atomic mass unit	1 u $= 1.66 \cdot 10^{-27}$ kg $= 931$ MeV/c^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J $\cdot$ s $= 4.41 \cdot 10^{-15}$ eV $\cdot$ s
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(N $\cdot$ m^2)
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}$ (N $\cdot$ m^2)/C^2
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}$ W/m^2/K^4

T1: Levitation

Levitation is a widely researched area in physics with wide applications in real life. Commercial high speed trains use magnetic levitation to transport passengers and the very physics of aerodynamics helps planes fly in the sky. Although the many uses of levitation physics apply on macroscopic dimensions, this problem will also analyze the various applications of levitation on the microscopic scale.

Optical Tweezers

An optical tweezer (OT) is a device which uses tightly focused laser beams to trap an object in all three spacial dimensions ($x-y-z$). Consider using an OT to trap a dielectric polystyrene nanosphere with mass m, radius R, and relative dielectric constant ε_r. A laser beam is directed to the vertical z-direction (the laser beam is similar to a monochromatic light wave propagating a sparse medium.) The laser beam has a wavelength λ. The time averaged intensity in the z-direction due to light can be considered to follow a Gaussian distribution function:

$$I(\rho, z) = I_0 \left(\frac{W_0}{W(z)}\right)^2 \exp\left(\frac{-2\rho^2}{W(z)^2}\right)$$

where ρ is the distance from the center of the beam and W_0 is known as the waist size, or the measure of the beam size at the point of its focus. Here the waist length, in general, follows $W(z) = W_0\sqrt{1+z^2/z_R^2}$ where $z_R = \pi W_0^2/\lambda$ denotes the Rayleigh length.

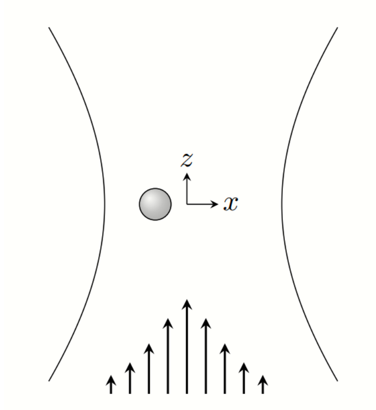

Figure 1: A nanosphere placed off center in a Gaussian beam.

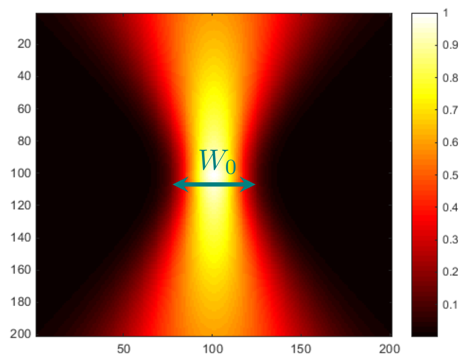

Figure 2: A graphic of the intensity distribution in a Gaussian beam.

The OT traps particles via three different forces: **scattering forces** created by the change in momentum of light scattered or absorbed by a particle; **gradient forces** due to the polarization of the particle created by the strong electric fields of the laser beam; and, **radiation forces** produced by an accelerating charge. The total power radiated by an oscillating electric dipole with dipole moment p_0 at frequency ω will be $P_R = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$, where c is the speed of light.

1. **(6 pts.)** In the Rayleigh spectrum, the nanosphere's size is such that $\lambda \gg R$. Find the oscillation frequency Ω and equilibrium position of the nanosphere when slightly displaced a distance $d \ll W_0$ in the x-direction. Neglect the scattering forces in this part.

In the Mie spectrum, the particle size is no longer negligible such as in part A, and follows $R \gtrsim \lambda$[1]. As a result, a non-homogenous electric field is incident upon the sphere and scattering forces are no longer neglectable. Parts B and C will investigate the nanosphere in the Mie spectrum.

[1]The order of magnitude of R is greater than λ.

2. **(5 pts.)** Determine the scattering force and torque on the nanosphere as a function of a distance $x \ll W_0$ from its origin. We can consider $x \sim R$. Assume that 100% of light is transmitted for simplicity.[2] The index of refraction of the nanosphere is n while the index of refraction of the medium is m. You may express your answer as an integral if needed.

3. **(2 pts.)** In a simplified model, the torque acting on the nanosphere can be represented as $\boldsymbol{\tau} = \kappa \boldsymbol{\omega}$ where κ is a numerical constant and $\boldsymbol{\omega}$ is the angular velocity of the nanosphere. At $t = 0$, the nanosphere is stationary with an angular velocity ω_0 and the OT is turned on. After a time T, the OT is turned off with the nanosphere left floating in the medium again. Determine the angular velocity of the nanosphere after a time t passes where t is the time of the entire process. It can be expressed as a piecewise function.

Acoustic Levitation

Objects can also be trapped via sound waves. Consider the simplest way to model sound waves; that is, in one dimension. A cylindrical tube of length L_0 with ambient temperature and pressure T_0 and P_0 is fixed with a piston at one end[3] of cross-sectional area S that moves periodically as $x(t) = A\cos(2\pi f t)$ where $A \ll L_0$ is the amplitude of the piston and f is the frequency of the process. The tube contains n monatomic particles of mass m per unit volume. As the piston moves back and forth, the air in the tube compresses or expands (rarefies) in the tube as a sound wave. The molecules within the tube move back or forth parallel to their equilibrium position as the air travels within the piston. Neglect any viscous or turbulent friction within the pipe.

4. **(6 pts.)** What is the average power required to move the piston? Consider the limits of $f \gg c_s/A$ and $f \ll c_s/A$ where c_s is the speed of the sound wave.

In sound waves, the density perturbations are very small, so it can be assumed that $\Delta\rho \ll \rho_0$[4]. where ρ_0 is the original density of the pipe. Furthermore, the wavelength of the sound waves are much larger than the mean free path of the gas molecules. The sound wave created by the oscillating piston moves at a speed c_s and dissipates the power given by the piston.

5. **(4 pts.)** As a result of compression, the air within the sound wave has a larger temperature $\Delta T \ll T_0$. Find the change in temperature ΔT. If you were unable to solve problem 4, express the average power to move the piston as P.

6. **(2 pts.)** Consider a small cylindrical object of radius $R < \sqrt{S}$ and width $h \ll R$ in the pipe where variations of pressure on the cylinder's surface are negligible. Determine the force F acting on the cylinder when the sound wave passes through it. If the pipe is placed on a vertical plane where gravity is present, qualitatively describe what location(s) the cylinder would levitate.

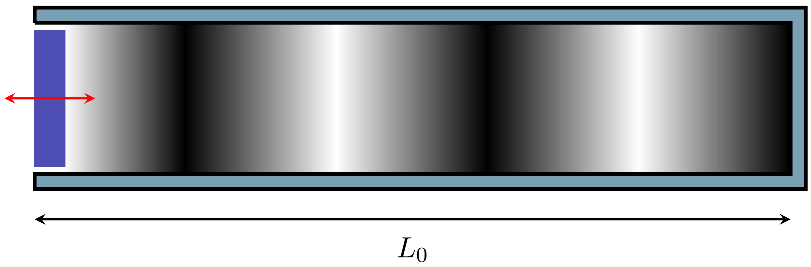

Figure 3: A visualization of how acoustic waves within the pipe are created via the oscillation of the piston.

[2]In reality, some light will be reflected due to Fresnel's equations.
[3]In some models, the piston can represent the moving cone of a loud speaker.
[4]You cannot use $\Delta\rho$ as a variable in this problem, but you can use ρ_0.

T2: Thomas Precession

Sucessive Transformations

In this section, we examine what happens when two successive Lorentz transformations are applied in non-parallel directions.

1. **(6 pts.)** Consider the three reference frames S_1, S_2, and S_3. Events as seen from frame S_i will be labeled with the space-time coordinates (x_i, y_i, z_i, t_i), for $i = 1, 2, 3$. All three frames coincide at $(0, 0, 0, 0)$. Suppose frame S_2 travels with velocity βc in the x_1-direction of frame S_1 and frame S_3 travels with velocity $\beta'_x c\hat{\mathbf{i}}_2 + \beta'_y c\hat{\mathbf{j}}_2$ with respect to frame S_2.

   Perform two successive Lorentz transformations: one expressing the S_3 coordinates in terms of the S_2 coordinates and another expressing the S_2 coordinates in terms of the S_1 coordinates. Then, as the final answer, express the S_3 coordinates in terms of the S_1 coordinates.

   Assume that $\beta' = (\beta'^2_x + \beta'^2_y)^{1/2} \ll \beta$ and work to first order.

2. **(5 pts.)** Now, find the velocity of S_3 in S_1 with the appropriate velocity addition. Perform a single Lorentz transformation to express the S_3 coordinates in terms of the S_1 coordinates. Once again, work to first order. The answer will not be the same as part 1.

3. **(3 pts.)** Show that your answer in Problem 1 differs from your answer in Problem 2 by a spatial rotation. In other words, two successive Lorentz transformations in non-parallel directions cannot be combined as one Lorentz transformation. Rather, they are the combination of one Lorentz transformation and one spatial rotation. Determine the magnitude and direction of this spatial rotation in the current setup.

   Then argue and explain why the answer in Problem 1 and not Problem 2 is the correct transformation.

Precession Frequency

In this section, we examine the precession of the electron's spin magnetic moment within the hydrogen atom.

Electrons possess an attribute known as *spin*. One can imagine the electron as being a small, spherical charged particle that is spinning on some axis. Although this mental picture is not physically correct, it is enough for us. From this mental picture, we can gather that the electron will possess a dipole moment and angular momentum *intrinsic* to itself and not induced by any orbital motion. Respectively, these are known as the *spin magnetic moment* and the *spin angular momentum* that behave, in our model, just as their classical counterparts do. These two vector quantities, $\boldsymbol{\mu}$ and $\mathbf{L}$ respectively, are related by

$$\boldsymbol{\mu} \simeq -\frac{e}{m_e}\mathbf{L},$$

which is determined by experiment.

In this section, we will use the Bohr model of the hydrogen atom, where the electron circles the proton at some radius r, pulled in by the Coulomb force.

4. **(3 pts.)** In the laboratory frame, the electron orbits the proton with some velocity v in the x-y plane. Now switch to the instantaneous rest frame of the electron, where the proton moves with speed v relative to the stationary electron. The moving proton will then induce some magnetic field at the electron's location.

   Suppose the spin angular momentum of the electron points in some direction other than the direction of the magnetic field. Because the electron also possesses a spin magnetic moment, it will precess as a result of the torque done on it. Find the angular frequency of the precession of the electron's spin in terms of r and whatever fundamental constants.

   Ignore any relativistic effects.

5. **(6 pts.)** This problem will use the answer from Part 1. Once again, assume that the electron is orbiting in the x-y plane.

   Consider the instantaneous rest frame of the electron at some time t, S_2. Also consider the instantaneous rest frame of the electron at some time $t + \mathrm{d}t$, S_3. Relative to the laboratory frame, S_1, S_2 will have velocity $\mathbf{v}$. S_3 will have a velocity $\mathrm{d}\mathbf{v}$ relative to S_2, but $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$ won't be parallel.

   The result from Part 1 tells us that S_3 will experience an infinitesimal rotation in this time $\mathrm{d}t$ with respect to the laboratory frame. Since the electron is continuously accelerating, its rest frame, S_3, must then rotate continuously relative to the lab frame. Assume that the spin of the electron always points in the same direction in its rest frame. If the spin is not pointing in the z direction, find the angular frequency of the spin's precession in terms of r and whatever fundamental constants. Ignore the effects of the previous problem.

   Note: although relativistic effects cannot be ignored, you can assume that the electron's velocity v is not comparable to the speed of light in the calculation.

6. **(2 pts.)** Combine your answers from parts 4 and 5 and find the relativistically correct angular frequency of the precession of the electron's spin.

T3: Moving Media

Interesting phenomena can arise in situations where there are 2 media that are moving with respect to each other. In particular, objects can move at much faster speeds than the relative speeds of the media, without using any energy. In this problem, we explore two such examples of this effect.

Moving Cylinders

Suppose we have three cylinders, two small cylinders and a large cylinder, of radii r and R. The frictionless pivots (centers) of the cylinders are attached to a massless triangular frame, such that the large cylinder is in contact with the two small cylinders but the two small cylinders are not touching each other. The small cylinders each have a thin groove along their circumferences (which does not affect the moment of inertia significantly), so that the large cylinder makes contact with the small cylinder at a point with radial distance αr from the center of the small cylinder. The axes of all cylinders are perpendicular to the plane of the triangular frame. The system is placed on a level ground and a long flat horizontal board is put on top of the large cylinder, with the two small cylinders touching the ground (making contact at their outer edge with radial distance r, not αr). Assume that the friction due to contact between all surfaces is large enough to prevent any slipping.

1. **(4 pts.)** The board is moved with speed v in a direction perpendicular to the axes of the cylinders. Find the speed of the cylinder system.

2. **(5 pts.)** The mass of the small and large cylinders are m and M, respectively. The mass of the board is m'. If at a moment in time the board is pushed with speed v and acceleration a, find the power P required to push the board. Assume the cylinders have uniform mass distribution.

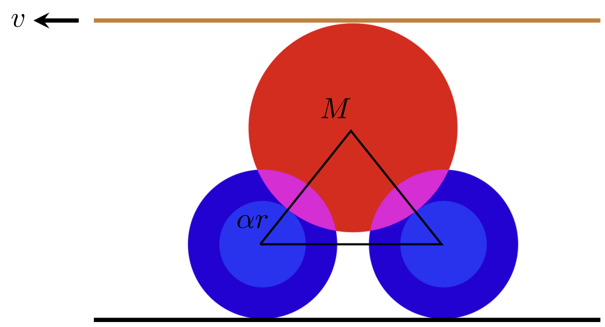

Figure 4: A visual of the three cylinder setup.

Windsurfing

In windsurfing, it is possible to sail faster than the wind without using any energy. Suppose we have a sailboat moving on a large, motionless body of water. The air of density ρ is moving at a speed v uniformly in one direction. If the sailboat is pointed in a certain direction and moves in that direction with velocity $\mathbf{u}$, the drag force from the water $\mathbf{F}$ satisfies $\mathbf{F} \cdot \mathbf{u} = -\gamma u^2$, where γ can be assumed to be a constant drag coefficient.

3. **(10 pts.)** If the wind is moving in the $\hat{x}$ direction, what is the maximum possible sustainable x-component of velocity for the sailboat? Assume that the sailboat can neither generate nor store energy in its interaction with the air. Also, the effective cross-sectional area of the sail is A (this is the component of cross-sectional area that is perpendicular to the wind in the reference frame of the sailboat).

4. **(3 pts.)** What is the power dissipated due to the interaction with the water?

5. **(3 pts.)** It seems that the law of conservation of energy is being violated, as the speed of the sailboat isn't changing despite heat generation in the water. Explain why energy is still conserved.

T4: Missing Energy

Part A

1. **(3 pts.)** Consider a simple circuit with two parallel-plate capacitors of capacitance C_1 and C_2 connected to each other using purely conducting wires and a switch. One of the capacitors is initially charged to a voltage V_0, while the other one is completely uncharged. The circuit is kept in a square shaped figure of side length ℓ throughout the problem, while the diameter of the conducting wires is D. Find the initial total energy of the circuit when the switch is open, given by E_0, and a sufficiently long time after the switch is closed, given by E_∞. Calculate the remaining energy $E_\Delta = E_0 - E_\infty$. What is E_Δ for the case $C_2 \to \infty$?

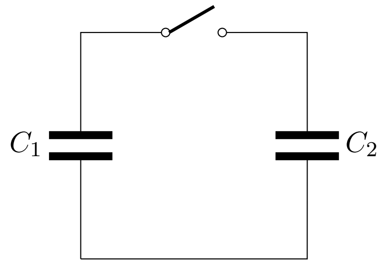

Figure 5: The two parallel plate capacitor-switch circuit.

It seems odd for there to be a difference in energy as the circuit is a closed system. Three young scientists Fermi, Jackson, and Feynman have created different theories to find and verify the correct source of this missing energy.

Thermal Losses: Fermi

To investigate the cause of this missing energy, Fermi assumes that there must be an ohmic resistive load r and a self-inductance L in the circuit responsible for E_Δ.

2. **(3 pts.)** Find the current in the circuit $I(t)$ as a function of time and E_Δ for the circuit.

3. **(1 pt.)** For small values of r, find the oscillation frequency Ω of $I(t)$.

4. **(1 pt.)** For $L = 0$, can Fermi's reasoning be correct for any value of r? If yes, what is this value of r?

Dipole Radiation Losses: Jackson

Jackson believes that the missing energy is dissipated in the form of dipole radiation losses due to the charges accelerating. He assumes that the electric dipole moment of the system remains constant during the process, but the magnetic moment is allowed to vary. Thus, he seeks to determine the maximal possible radiation losses. For this he uses Larmor's formula, which states that for small velocities relative to the speed of light c, total power radiated which the radiation power is defined as:

$$P_r = \frac{\ddot{m}^2}{4\pi\epsilon_0 c^3}$$

where m is the magnetic moment of the circuit as a function of time. Ignore all relativistic effects and the possible charge accumulation in the wires compared to that on the capacitor plates. Moreover, note that he does not assume any resistance or self-inductance in the circuit in his model.

5. **(4 pts.)** Find the total energy dissipation E_r due to this radiation.

6. **(1 pt.)** For what value of time interval $\Delta\tau$ taken by the charges to move from one capacitor to another, can Jackson's theory be reasoned true?

Kinetic Energy: Feynman

Feynman has the following hypothesis:

> The missing energy goes into the kinetic energy of the charge carriers going from C_1 to C_2. Assume that the mean free path of collisions of the carriers is $\lambda > 2\ell$.

7. **(4 pts.)** Find the total kinetic energy ΔK gained by the carriers during a total charge transfer from C_1 to C_2.

8. **(1 pt.)** Could this be a valid hypothesis to explain the cause of the missing energy? When the charges get completely deposited on the plates of C_2, what happens to this kinetic energy?

Part B

Instead of charging one capacitor using the other, we take an ideal parallel-plate capacitor such that the surface charge density $\pm\sigma$ on its plates is uniform throughout both plates, and that the charges are 'fixed' to the surface as the plate expands. The dimensions of the plates are a, b and the plate separation distance is d. The plate is now stretched quasi-statically by a factor of κ in one of the dimensions such that the dimensions of the capacitor plates are now $\kappa a, b$ but the plate separation remains d.

1. **(6 pts.)** Calculate the work $\mathrm{d}W$ done during stretching the plates of this capacitor. Also write a simplified form of this expression for $d \ll \kappa a, b$.

2. **(1 pt.)** In this case, there are no resistive loads, and since the process of plate expansion is quasi-static, there is no gain in kinetic energy of the charge carriers. Where does the energy disappear in this case?