2020 Online Physics Olympiad (OPhO): Open Contest

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.81$ m/s^2 in this contest. See the constants sheet on the following page for other constants.
- This test contains 55 short answer questions. Each problem will have three possible attempts.
- The weight of each question depends on our scoring system found here. Put simply, the later questions are worth more, and the overall amount of points from a certain question decreases with the number of attempts and days that you take to solve a problem as well as the number of teams who solve it. This means that your score decreases with the number of tries and days you take to solve a given problem.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should contain at least **three** significant figures, unless otherwise specified. All answers within the 1% range will be accepted.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used. Technology and computer algebra systems like Wolfram Alpha or the one in the TI nSpire will not be needed. Attempts to use these tools will be classified as cheating.
- You are *allowed* to use Wikipedia or books in this exam. Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions.
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt) unless otherwise specified. Please answer all questions in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put letters in your answer on the submission portal! If your answer is "x meters" , input only the value x into the submission portal.
- **Do not communicate information to anyone else apart from your team-members before the exam ends on May 29, 2020 at 11:59 PM UTC**.

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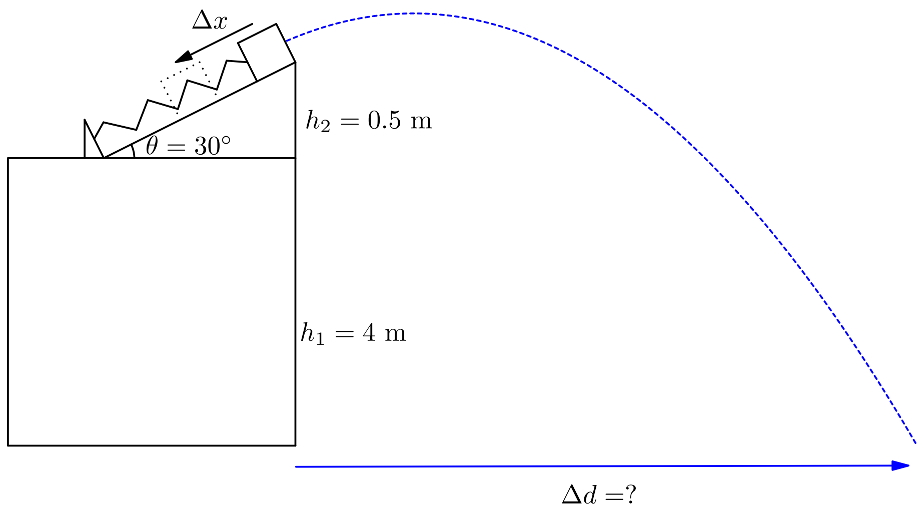
List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}\ \text{mol}^{-1}$
- Universal gas constant, $R = 8.31\ \text{J}/(\text{mol} \cdot \text{K})$
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$
- 1 electron volt, $1\ \text{eV} = 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\text{N} \cdot \text{m}^2)/\text{kg}^2$$
- Acceleration due to gravity, $g = 9.81\ \text{m/s}^2$
- 1 unified atomic mass unit,
$$1\ \text{u} = 1.66 \cdot 10^{-27}\ \text{kg} = 931\ \text{MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \text{J} \cdot \text{s} = 4.41 \cdot 10^{-15}\ \text{eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \text{C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\text{N} \cdot \text{m}^2)/\text{C}^2$$
- Permeability of free space, $\mu_0 = 4\pi \cdot 10^{-7}\ \text{T} \cdot \text{m/A}$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$$
- 1 atmospheric pressure,
$$1\ \text{atm} = 1.01 \cdot 10^{5}\ \text{N/m}^2 = 1.01 \cdot 10^{5}\ \text{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}\ \text{m} \cdot \text{K}$
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2/\text{K}^4$$

Problems

Pr 1. Angry Birds

A quarantined physics student decides to perform an experiment to land a small box of mass $m = 60$ g onto the center of a target a distance Δd away. The student puts the box on a top of a frictionless ramp with height $h_2 = 0.5$ m that is angled $\theta = 30°$ to the horizontal on a table that is $h_1 = 4$ m above the floor. If the student pushes the spring with spring constant $k = 6.5$ N/m down by $\Delta x = 0.3$ m compared to its rest length and lands the box exactly on the target, what is Δd? Answer in meters. You may assume friction is negligible.



Solution: By conservation of energy, we have that

$$\frac{1}{2}kx^2 = mgx\sin\theta + \frac{1}{2}mv^2 \implies v = \sqrt{\frac{k}{m}x^2 - 2gx\sin\theta}.$$

By kinematic formulae for motion with constant acceleration,

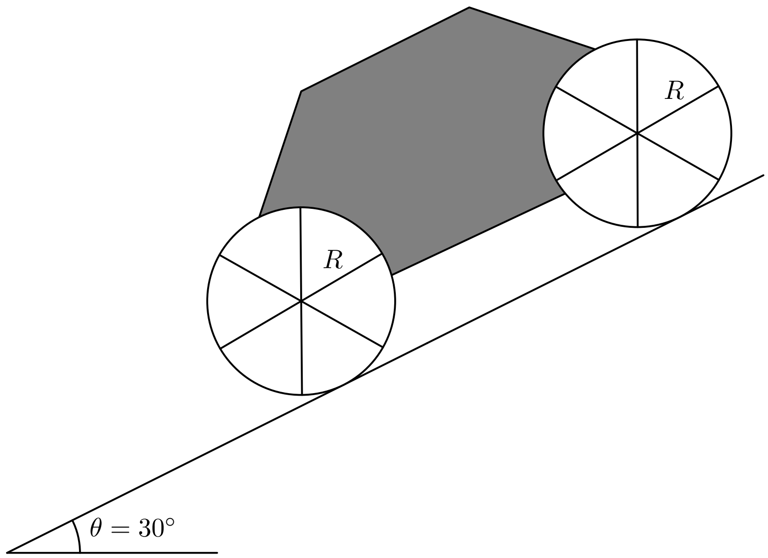
$$-4.5 \text{ m} = v\sin\theta t - \frac{1}{2}gt^2.$$

Solving for t through the quadratic, $t = 1.099$ s and the velocity is $v = 2.60$ m/s. Therefore, the distance, $\Delta d = v\cos\theta t = \boxed{2.47 \text{ m}}$.

Pr 2. The Wheels on the Monster Truck go Round and Round

A wooden bus of mass $M = 20,000$ kg (M represents the mass excluding the wheels) is on a ramp with angle $30°$. Each of the four wheels is composed of a ring of mass $\frac{M}{2}$ and radius $R = 1$ m and 6 evenly spaced spokes of mass $\frac{M}{6}$ and length R. All components of the truck have a uniform density. Find the acceleration of the bus down the ramp assuming that it rolls without slipping.

Answer in m/s^2.



Solution: The moment of inertia of each wheel can be thought of as a superposition of the six spokes and a ring. Therefore, we get:

$$I_{\text{wheel}} = \frac{M}{2}R^2 + 6\left(\frac{1}{3}\frac{M}{6}R^2\right) = \frac{5}{6}MR^2.$$

The moment of inertia of four wheels is:

$$I_{\text{bus}} = 4\left(\frac{5}{6}MR^2\right) = \frac{10}{3}MR^2.$$

The total mass of the bus is

$$m = M + 4\left(\frac{M}{2} + 6\cdot\frac{M}{6}\right) = 7M.$$

Using Newton's second law down the ramp,

$$7Ma = 7Mg\sin\theta - 4f$$

if f is the friction at each wheel, and the torque balance on each wheel is:

$$\frac{5}{6}MR^2\alpha = fR.$$

Letting $a = \alpha r$ for the no slip condition, we can solve for f to be:

$$f = \frac{5}{6}Ma.$$

so our force balance equation becomes:

$$7Ma = 7Mg\sin\theta - \frac{10}{3}Ma \implies a = \frac{21}{31}g\sin\theta = \boxed{3.32\ \text{m/s}^2}$$

Pr 3. District 12

In an old coal factory, a conveyor belt will move at a constant velocity of 20.3 m/s and can deliver a maximum power of 15 MW. Each wheel in the conveyor belt has a diameter of 2 m. However a changing demand has pushed the coal factory to fill their coal hoppers with a different material with a certain constant specific density. These "coal" hoppers have been modified to deliver a constant 18 m^3s^{-1} of the new material to the conveyor belt. Assume that the kinetic and static friction are the same and that there is no slippage. What is the maximum density of the material?

Solution: The maximal force the convey belt can provide to a particle is:

$$F = \frac{P}{v}$$

The conveyor belt must provide an impulse to the particles to have a momentum of $p = mv$, where m is the mass of the particle and v is the velocity.

$$F = \frac{dp}{dt}$$

where $\frac{dp}{dt}$ is:

$$\rho \dot{V} v$$

Solving for for the maximum density we get:

$$\rho = \frac{P}{\dot{V} v^2}$$

$$\rho = \boxed{2022.2\ \frac{\text{kg}}{\text{m}^3}}$$

Pr 4. Neutrino Party

Neutrinos are extremely light particles and rarely interact with matter. The Sun emits neutrinos, each with an energy of 8×10^{-14} J and reaches a flux density of 10^{11} neutrinos/(s cm^2) at Earth's surface.

In the movie *2012*, neutrinos have mutated and now are completely absorbed by the Earth's inner core, heating it up. Model the inner core as a sphere of radius 1200 km, density 12.8 g/cm^3, and a specific heat of 0.400J/g K. The time scale, in seconds, that it will take to heat up the inner core by 1°C is $t = 1 \times 10^N$ where N is an integer. What is the value of N?

Solution: The cross sectional area is πr^2, so the incoming power generated by the neutrinos is:

$$P = \pi r^2 E \Phi$$

where E is the energy of each neutrino and Φ is the flux density. We want to cause a change in energy of:

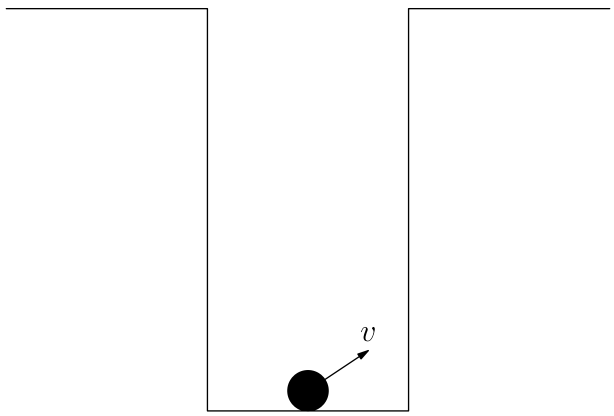
$$\Delta Q = mc\Delta T = \rho \frac{4}{3} \pi r^3 c \Delta T$$

which can be accomplished in a time:

$$Pt = \Delta Q \implies t = \frac{\rho(4\pi r^3)c\Delta T}{3\pi r^2 E\Phi} = \frac{4\rho r c \Delta T}{3E\Phi} = \boxed{1 \times 10^{14}\ \text{s}}$$

Pr 5. Quarantine Secrets

A ball is situated at the midpoint of the bottom of a rectangular ditch with width 1 m. It is shot at a velocity $v = 5$ m/s at an angle of 30° relative to the horizontal. How many times does the ball collide with the walls of the ditch until it hits the bottom of the ditch again? Assume all collisions to be elastic and that the ball never flies out of the ditch.



Solution: We use the idea of mirroring the walls of the ditch. We can then draw out the normal path of the projectile and find the number of intersections the projectile makes with the mirror walls. The total time of the projectile to travel a path is given by

$$t = \frac{2v_0 \sin\theta}{g}.$$

The projectile will cover a horizontal distance

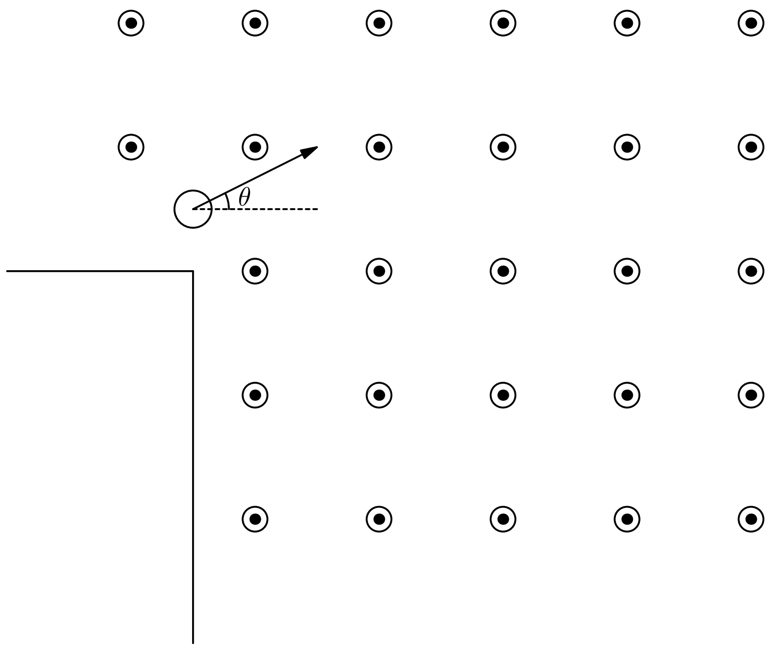
$$\left(N + \frac{1}{2}\right) a = v\cos\theta t \implies N = \frac{v}{a}\cos\theta t - \frac{1}{2}.$$

Taking the ceiling of this gives:

$$N = \left\lceil \frac{R - a/2}{a} \right\rceil = \boxed{2}.$$

Pr 6. Planetary Proton

Professor Proton has discovered a new planet on one of his planetary expeditions. He wants to measure the magnetic field of the planet he has found. Professor Proton has brought all the necessary equipment required to carry out the following experiment. A proton is launched off a large cliff at a non-relativistic speed v and an angle $\theta = 30^\circ$ with respect to the horizontal plane at the magnetic equator of a distant planet. The magnetic field acting perpendicularly on the particle can be assumed to be perfectly horizontal and coming out of the page, as shown in the diagram. How strong is the magnetic field at the magnetic equator of this planet if the period of oscillation of v_x is 4.94×10^{-4} s? Write your answer in terms of μT (micro-Teslas).



Solution: The separate time-dependent equations for the x and y components of velocity are

$$\dot{v_y} = \frac{-qv_xB - mg}{m}$$

$$\dot{v_x} = \frac{qv_yB}{m}$$

Taking the derivative of the second equation:

$$\ddot{v_x} = \frac{q\dot{v_y}B}{m}$$

Plugging,

$$\dot{v_y} = \frac{\ddot{v_x}m}{qB}$$

into

$$\dot{v_y} = \frac{-qv_xB - mg}{m}$$

we get a differential equation that has a sinusoidal solution. The angular frequency is therefore: $\omega = \frac{qB}{m}$. Solving for period B we get that

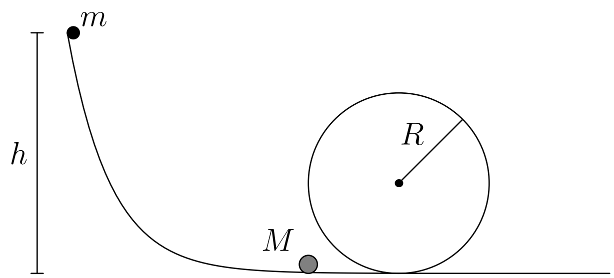
$$B = \frac{\omega m}{q}$$

$$B = \frac{\frac{2\pi}{T}m}{q}$$

$$B = \boxed{132.76\ \mu\text{T}}$$

Pr 7. Angel Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball (of negligible size) of mass $M = 5$ kg.



Let h be the minimum height that m was dropped such that M would be able to move all the way around the loop. The coefficient of restitution for this collision is given as $e = \frac{1}{2}$.

Now consider a different scenario. Assume that the balls can now collide perfectly inelastically, which means that they stick to each other instantaneously after collision for the rest of the motion. If m was dropped from a height $3R$, find the minimum value of $\frac{m}{M}$ such that the combined mass can fully move all the way around the loop. Let this minimum value be k. Compute $\alpha = \frac{k^2}{h^2}$. (Note that this question is *only* asking for α but you need to find h to find α). Assume the balls are point masses (neglect rotational effects).

Solution: The velocity of m when it gets to the bottom of the track will be given by

$$v_b = \sqrt{2gh}.$$

Claim: The velocity of M after collision will be given by $v_M = \frac{(1+e)m}{m+M}\sqrt{2gh}$.

Proof: Conservation of momentum before and after the collision is expressed by:

$$mv_b \;=\; mv_m + Mv_M$$

By coefficient of restitution,

$$v_M - v_m = ev_b$$

These equations may be solved directly to find v_m, v_M to give

$$v_M = \frac{(1+e)m}{m+M}v_b \quad \square$$

Next, when the objects get to the top of the loop, conservation of energy gives the speed of M when it gets to the top as

$$\frac{1}{2}v_M^2 = \frac{1}{2}v_t^2 + 2gR.$$

At the top of the loop, M must at least have an acceleration g to maintain circular motion, and thus

$$\frac{v_t^2}{R} = g \implies v_t = \sqrt{gR}.$$

Substituting these results into our conservation equation gives us

$$\frac{(1+e)^2m^2}{(m+M)^2}(\sqrt{2gh})^2 = gR + 4gR$$
$$\frac{2(1+e)^2m^2gh}{(m+M)^2} = 5gR$$
$$h = \frac{5(m+M)^2R}{2(1+e)^2m^2}$$
$$= 6.805 \text{ m}$$

In the second scenario, conservation of momentum before and after the collision gives:

$$mv_b = (M+m)v_f \implies v_f = \frac{mv_b}{M+m}$$

The same conservation of energy formula as in the first scenario yields

$$\frac{m^2}{(M+m)^2}\left(\sqrt{2gh}\right)^2 = 5gR$$
$$\frac{2m^2gh}{(M+m)^2} = 5gR$$
$$\frac{6m^2gR}{(M+m)^2} = 5gR$$
$$\left(\frac{k}{k+1}\right)^2 = \frac{5}{6}$$
$$\frac{k}{k+1} = \sqrt{\frac{5}{6}}$$
$$k = \frac{\sqrt{5}}{\sqrt{6}-\sqrt{5}}$$
$$= 10.477$$

Thus,

$$\alpha = \frac{k^2}{h^2} = \boxed{2.37 \text{ m}^2}$$

Pr 8. Wannabe Twoset

Eddie is experimenting with his sister's violin. Allow the "A" string of his sister's violin have an ultimate tensile strength σ_1. He tunes a string up to its highest possible frequency f_1 before it breaks. He then builds an exact copy of the violin, where all lengths have been increased by a factor of $\sqrt{2}$ and tunes the same string again to its highest possible frequency f_2. What is f_2/f_1? The density of the string does not change.

Note: The ultimate tensile strength is maximum amount of stress an object can endure without breaking. Stress is defined as $\frac{F}{A}$, or force per unit area.

Solution: We note from a simple dimensional analysis that the angular frequency of the string ω will consist of the tension T, the length of the string L and the mass of the string m.

$$T = [MLT^{-2}]$$
$$L = [L]$$
$$m = [M]$$
$$\omega = [T^{-1}]$$

Therefore, by rearranging, we find that

$$\omega = T^\alpha L^\beta m^\gamma$$
$$[T^{-1}] = [MLT^{-2}]^\alpha [L]^\beta [M]^\gamma$$

Distributing the exponents, and rearranging gives us

$$T^{-1} = M^{\alpha+\gamma} L^{\alpha+\beta} T^{-2\alpha}$$

We now have three equations

$$\alpha + \gamma = 0$$
$$\alpha + \beta = 0$$
$$-2\alpha = -1$$

From here, we find that $\alpha = 1/2$. Substituting this into the first equation gives us

$$1/2 + \gamma = 0 \implies \gamma = -1/2$$

then substituting α into the second equation gives us

$$1/2 + \beta = 0 \implies \beta = -1/2$$

We now find that the angular frequency is given by

$$\omega = A\sqrt{\frac{T}{Lm}}$$

where A is an arbitrary constant. Noting that $\omega = 2\pi f$, we find that

$$f = \frac{A}{2\pi}\sqrt{\frac{T}{Lm}}.$$

From this analysis, we can then see that $f_2/f_1 = \sqrt{2}/2 \approx \boxed{0.707}$.

Pr 9. Waterhorse or Flyinghorse

A one horsepower propeller powered by a battery and is used to propel a small boat initially at rest. You have two options:

1. Put the propeller on top of the boat and push on the air with an initial force F_1
2. Put the propeller underwater and push on the water with an initial force F_2.

The density of water is 997 kg/m^3 while the density of air is 1.23 kg/m^3. Assume that the force is both cases is dependent upon only the density of the medium, the surface area of the propeller, and the power delivered by the battery. What is F_2/F_1? You may assume (unrealistically) the efficiency of the propeller does not change. Round to the nearest tenths.

Solution: The force exerted on the fluid is roughly proportional to the change in momentum with respect to time:

$$F = \frac{dp}{dt} = v\frac{dm}{dt} = v\frac{d}{dt}(\rho A x) = \rho A v^2$$

It is kept at a constant power $P = Fv$, which can allow us to solve for the speed v of the propellers.

$$P = \rho A v^3 \implies v = \left(\frac{P}{\rho A}\right)^{1/3}$$

so the force is given by:

$$F = \rho A \left(\frac{P}{\rho A}\right)^{2/3} \implies F \propto \rho^{1/3}$$

Therefore:

$$F_2/F_1 = (997/1.23)^{1/3} = \boxed{9.26} \text{ times}$$

Pr 10. Charlie And The Chocolate Factory

A professional pastry chef is making a sweet which consists of 3 sheets of chocolate. The chef leaves a gap with width $d_1 = 0.1$ m between the top and middle layers and fills it with a chocolate syrup with uniform viscosity $\eta_1 = 10\,\mathrm{Pa\cdot s}$ and a gap with width $d_2 = 0.2$ m between the middle and bottom sheet and fills it with caramel with uniform viscosity $\eta_2 = 15\,\mathrm{Pa\cdot s}$. If the chef pulls the top sheet with a velocity $2\,\mathrm{m/s}$ horizontally, at what speed must he push the bottom sheet horizontally such that the middle sheet remains stationary initially? Ignore the weight of the pastry sheets throughout the problem and the assume the sheets are equally sized.

Note: Shear stress is governed by the equation $\tau = \eta \times$ rate of strain.

Solution: The plates are equal sizes so all we have to do is simply balance the shear stresses which act in opposing directions on the middle plate:

$$\tau_1 = \tau_2$$

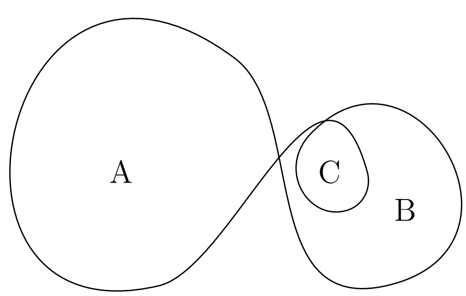
$$\eta_1 \cdot \frac{v_1}{d_1} = \eta_2 \cdot \frac{v_2}{d_2}$$

$$10 \cdot \frac{2}{0.1} = 15\frac{v}{0.2}$$

$$v = \boxed{2.667 \text{ m/s}}.$$

Pr 11. Loopy Wire

The following diagram depicts a single wire that is bent into the shape below. The circuit is placed in a magnetic field pointing out of the page, uniformly increasing at the rate $\frac{dB}{dt} = 2.34$ T/s. Calculate the magnitude of induced electromotive force in the wire, in terms of the following labelled areas (m^2). Note that B is non-inclusive of C and that $A = 4.23$, $B = 2.74$, and $C = 0.34$.



Solution: Without loss of generality, let the current around A flow in the counterclockwise direction and let the flux through A be positive. Note that the current will flow in the clockwise direction in C and around B, the area enclosed by the loop is $B + C$. The flux will be negative here. Therefore, the total flux is then proportional to:

$$\Phi \propto A - B - 2C = 0.81$$

and the magnitude of the induced electromotive force is:

$$\varepsilon = (A - B - 2C)\frac{dB}{dt} = \boxed{1.9\ \mathrm{T\,m^2\,s^{-1}}}.$$

The following information applies to the next two problems. A magnetic field is located within a region enclosed by an elliptical island with semi-minor axis of $a = 100$ m and semi-major axis of $b = 200$ m. A car carrying charge $+Q = 1.5$ C drives on the boundary of the island at a constant speed of $v = 5$ m/s and has mass $m = 2000$ kg. Any dimensions of the car can be assumed to be much smaller than the dimensions of the island. Ignore any contributions to the magnetic field from the moving car and assume that the car has enough traction to continue driving in its elliptical path.

Let the center of the island be located at the point $(0, 0)$ while the semi major and semi minor axes lie on the x and y-axes, respectively.

On this island, the magnetic field varies as a function of x and y: $B(x, y) = k_b e^{c_b xy}\hat{z}$ (pointing in the upward direction, perpendicular to the island plane in the positive z-direction). The constant $c_b = 10^{-4}$ m^{-2} and the constant $k_b = 2.1$ μT

Pr 12. Journey 2: The Magnetic Island 1

At what point on the island is the force from the magnetic field a maximum? Write the distance of this point from the x-axis in metres.

Pr 13. Journey 2: The Magnetic Island 2

Assuming no slipping, what is the magnitude of the net force on the car at the point of the maximum magnetic field? (Answer in Newtons.)

Solution:

(12) To find a minimum or maximum, the gradient of the constraint function $f(x, y) = \frac{x^2}{b^2} + \frac{y^2}{a^2} - 1$ and the gradient of the B field function should be scalar multiples of each other.

$$\frac{2x}{b^2}\mu = c_b y e^{xy}$$

$$\frac{2y}{a^2}\mu = c_b x e^{xy}$$

Solving the two equations, we get that a maximum point (x, y) is of the form $(\frac{b}{\sqrt{2}}, \frac{a}{\sqrt{2}})$ or $(-\frac{b}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$. The distance from the y-axis is thus $\frac{a}{\sqrt{2}} = \boxed{70.7 \text{ m}}$.

(13) The net force acting on the car must provide a_c. We have

$$r(x) = x\hat{i} + a\sqrt{1 - \frac{x^2}{b^2}}\hat{j}$$

and we simply have to find the radius of curvature of this function. This is given by

$$R = \frac{[1 + (y'(x))^2]^{\frac{3}{2}}}{|y''(x)|}.$$

We can evaluate this to find

$$R = \frac{(a^2 + b^2)^{3/2}}{2\sqrt{2}ab}.$$

The total force acting on the car is $m\frac{v^2}{R} = \boxed{252.98 \text{ N}}$.

Pr 14. Tuning Outside

Inside a laboratory at room temperature, a steel tuning fork in the shape of a U is struck and begins to vibrate at $f = 426$ Hz. The tuning fork is then brought outside where it is 10°C hotter and the experiment is performed again. What is the change in frequency, Δf of the tuning fork? (A positive value will indicate an increase in frequency, and a negative value will indicate a decrease.)

Note: The linear thermal coefficient of expansion for steel is $\alpha = 1.5 \times 10^{-5}\ \mathrm{K}^{-1}$ and you may assume the expansion is isotropic and linear. When the steel bends, there is a restoring torque $\tau = -\kappa\theta$ such that $\kappa \equiv GJ$ where $G = 77$ GPa is constant and J depends on the geometry and dimensions of the cross-sectional area.

Solution: Note that κ has units of torque so dimensionally, J must be proportional to L^3. Therefore, we have:

$$\beta M L^2 \alpha \propto -L^3\theta \implies f \propto \sqrt{L}$$

Therefore, we have:

$$\frac{\Delta f}{f} = \frac{\Delta L}{2L}$$

Since $\frac{\Delta L}{L} = \alpha\Delta T$, this gives us:

$$\Delta f = \frac{1}{2} f\alpha\Delta T = \boxed{0.0320 \text{ Hz}}.$$

Pr 15. Too Much Potential

A large metal conducting sphere with radius 10 m at an initial potential of 0 and an infinite supply of smaller conducting spheres of radius 1 m and potential 10 V are placed into contact in such a way: the large metal conducting sphere is contacted with each smaller sphere one at a time. You may also assume the spheres are touched using a thin conducting wire that places the two spheres sufficiently far away from each other such that their own spherical charge symmetry is maintained. What is the least number of smaller spheres required to be touched with the larger sphere such that the potential of the larger sphere reaches 9 V? Assume that the charges distribute slowly and that the point of contact between the rod and the spheres is not a sharp point.

Solution: Let each sphere with radius 1 m have charge q. Note that each time the large metal conducting sphere is contacted with each of the smaller spheres, the potential is equalized between the two objects. The potential on a sphere is proportional to $\frac{q}{r}$, so the large conducting sphere must retain $\frac{10}{11}$ of the total charge after it is contacted with a smaller sphere. Furthermore, to reach 9 V, the required end charge on the sphere of radius 10 m is at least $9q$. Thus, we get a recursion for the charge of the large square Q in terms of the number of small spheres touched n.

$$Q(n+1) = (Q(n) + q) \cdot \frac{10}{11}$$

Inductively applying this recursion, we obtain

$$Q(n) = q\left[\left(\frac{10}{11}\right)^n + \cdots + \frac{10}{11}\right].$$

We can now sum this geometric series:

$$Q(n) = q\left(\frac{10}{11} \cdot \frac{\left(\frac{10}{11}\right)^n - 1}{\frac{10}{11} - 1}\right).$$

Thus, using $Q(n) \geq 9q$, we find that $\left(\frac{10}{11}\right)^n \leq 0.1$, which provides $n \geq 24.1588$, or $n = \boxed{25}$ as the answer.

Pr 16. Particle in the Box

During high speed motion in a strong electric field, a charged particle can ionize air molecules it collides with.

A charged particle of mass $m = 0.1$ kg and charge $q = 0.5\ \mu$C is located in the center of a cubical box. Each vertex of the box is fixed in space and has a charge of $Q = -4\ \mu$C. If the side length of the box is $l = 1.5$ m what minimum speed (parallel to an edge) should be given to the particle for it to exit the box (even if it's just momentarily)? Let the energy loss from Corona discharge and other radiation effects be $E = 0.00250$ J.

Solution: Conservation of energy gives:

$$T_i + U_i = T_f + U_f + E.$$

Solving for the initial potential energy gives

$$U_i = -8\frac{kqQ}{l\sqrt{3}/2} = -\frac{16kqQ}{\sqrt{3}l}.$$

And since the final kinetic energy is zero, the final potential energy is

$$U_f = -4\frac{kqQ}{l/\sqrt{2}} - 4\frac{kqQ}{l\sqrt{\frac{3}{2}}} = -\left(4\sqrt{2} + 4\sqrt{\frac{2}{3}}\right)\frac{kqQ}{l}$$

and thus solving for the initial kinetic energy:

$$\frac{1}{2}mv^2 = \left(-4\sqrt{2} - 4\sqrt{\frac{2}{3}} + \frac{16}{\sqrt{3}}\right)\frac{kqQ}{l} + E.$$

The final answer is $v = \boxed{0.354 \text{ m/s}}$.

Pr 17. Room of Mirrors 1

Max finds himself trapped in the center of a mirror walled equilateral triangular room. What minimum beam angle must his flashlight have so that any point of illumination in the room can be traced back to his flashlight with at most 1 bounce? (Answer in degrees.) Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant.

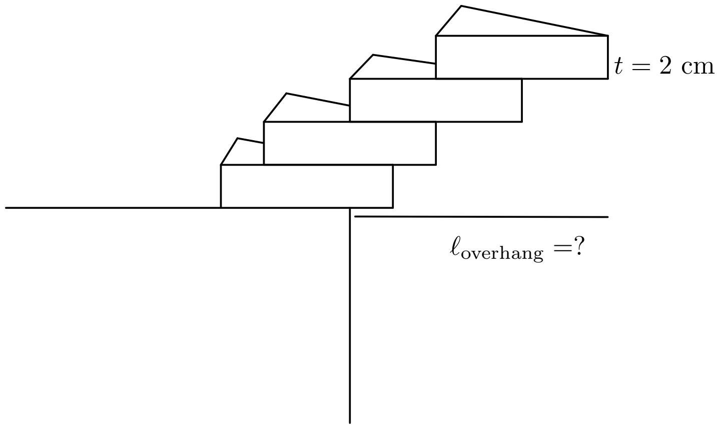
The point of illumination refers to any point in the room that is lit.

Solution: Each time light hits a mirror, we can reflect the entire equilateral triangle about that mirror and continue to trace the straight-line path of the light. For a maximum of 1 bounce, we can reflect our triangle about each of its initial sides. In order for the light to hit every part of the triangle (or an image of that part), by symmetry we require a $\boxed{120^\circ}$ angle. In this case, we would just shine the flashlight so that light directly reaches the entirety of one side, and reflections of light will fully reach the other two sides of the triangle.

Pr 18. Secret Society

For his art project, Weishaupt cut out $N = 20$ wooden equilateral triangular blocks with a side length of $\ell = 10$ cm and a thickness of $t = 2$ cm, each with the same mass and uniform density. He wishes to stack one on top of the other overhanging the edge of his table. In centimeters, what is the maximum overhang? Round to the nearest centimeter. A side view is shown below. Assume that all triangles are parallel to each other.

Note: This diagram is not to scale.



Solution: Let us consider $N = 1$ equilateral triangles. From inspection, we need to place the triangle such that the center of mass lies at the edge of the table. The maximum overhang in this case is $(1 - f)h$ where $h = \frac{\ell\sqrt{3}}{2}$ is the height of the triangle and $fh = \frac{h}{3}$ is the location of the center of mass.
If we wish to place a second triangle on top, we want to maximize the center of mass to be as far right as possible without the top block toppling. Placing the second block such that its center of mass is at the tip of the first triangle accomplishes this. However, the center of mass of the two triangles combined is now past the edge. Their center of mass is:

$$x_{\text{cm}} = \frac{fh + h}{2} = \frac{f + 1}{2}h$$

Thus the maximum overhang of the first block is now:

$$h - \frac{f + 1}{2}h = \frac{1 - f}{2}h$$

Now, we will place a third block such that it has the maximum overhang with respect to the top block and then shift the entire setup so that the center of mass of the system lies at the edge of the table. Following the same procedures, we find that the maximum overhang of the first block is:

$$\frac{1 - f}{3}h$$

The overhang of the top two blocks are $(1 - f)h$ and $\frac{1-f}{2}h$, unchanged from earlier. You can show via induction that the maximum overhang of the n^{th} block (counting from the top downwards) is:

$$\frac{1 - f}{n}h$$

so if there are 20 such blocks, then the total overhang (summing over all the blocks) is:

$$\sum_{k=0}^{20} \frac{1 - f}{k}h = (1 - f)hH_{20} = \frac{2}{3}\frac{\ell\sqrt{3}}{2}H_{20} = \frac{\ell\sqrt{3}}{3}H_{20} = 20.77 \text{ cm} \approx \boxed{21 \text{ cm}}$$

where H_N is the N^{th} harmonic number.

The following information applies to the following three problems. Kushal finds himself trapped in a large room with mirrors as walls. Being scared of the dark, he has a powerful flashlight to light the room. All references to "percent" refer to area. Since the room is large, assume the person is a point does not block light. Visualize the questions in a 2D setup. The floor/ceiling is irrelevant. *The point of illumination refers to any point in the room that is lit.*

Pr 19. Focus On That Not This! 1

What percent of a large circular room can be lit up using a flashlight with a 20 degree beam angle if Kushal stands in the center?

Pr 20. Focus On That Not This! 2

Kushal stands at a focus of an elliptical room with eccentricity 0.5 and semi major axis = 20 m. He points the flashlight along the semi-major axis away from the other focus. Find the ideal position where the torch can be placed to catch fire easily by the beam from the flashlight. What is the distance from this point to Kushal? Note that the torch cannot be at the same location as the flashlight. (Answer in metres.)

Pr 21. Focus On That Not This! 3

Now Kushal stands at a focus of the same elliptical room as in Problem 22. Determine the minimum percent of the elliptical room that can be lit up with a flashlight of beam angle 1 degree.

Solution:

(19) Each ray emitted follows a straight line, even when it is reflected since it originates from the center of a circle. Thus, the light rays in total trace out two circular sectors with an angle of $\theta = 20°$ each. Thus, the total percent of the room illuminated is:

$$f = \frac{2\theta}{360} = \boxed{11.1\%}.$$

(20) By the property of an ellipse, any light that is emitted from one focus and bounces off the sides will arrive at the other focus. The distance between the two foci is:

$$2ae = \boxed{20 \text{ m}}$$

(21) Let c be the distance between the foci and the center. The overlap area is: $\frac{1}{2}(2c)\left(\frac{c\theta}{6}\right) = \frac{c^2\theta}{6}$. PIE then gives:

$$A = \frac{1}{2}(a+c)^2\left(\frac{\theta}{3} + \frac{\theta}{9}\right) - \frac{c^2\theta}{6}$$

Letting $c = ae$ gives:

$$A = \frac{2}{9}a^2(1+e)^2\theta - \frac{a^2e^2\theta}{6} = a^2\theta\left(\frac{2}{9}(1+e)^2 - \frac{e^2}{6}\right)$$

the area of the ellipse is $\pi a^2\sqrt{1-e^2}$ so the fraction is:

$$f = \frac{\theta\left(\frac{2}{9}(1+e)^2 - \frac{e^2}{6}\right)}{\pi\sqrt{1-e^2}} \approx \boxed{0.294\%}$$

Pr 22. Two Star Crossed Lovers...

Two identical neutron stars with mass $m = 4 \times 10^{30}$ kg and radius 15 km are orbiting each other a distance $d = 700$ km away from each other (d refers to the initial distance between the cores of the neutron stars). Assume that they orbit as predicted by classical mechanics, except that they generate gravitational waves. The power dissipated through these waves is given by:

$$P = \frac{32G^4}{5}\left(\frac{m}{dc}\right)^5$$

How long does it take for the two stars to collide? Answer in seconds. *Note:* d is the distance between the *cores* of the stars.

Solution: Due to Virial theorem, we have:

$$K = -\frac{1}{2}U$$

so the total energy is:

$$E = U - \frac{1}{2}U = -\frac{Gm^2}{2R}$$

We know that the power dissipated gives the change in energy, or:

$$P = \frac{32G^4}{5}\left(\frac{m}{Rc}\right)^5 = \frac{d}{dt}\frac{Gm^2}{2R}$$

or:

$$\frac{32G^4}{5}\left(\frac{m}{Rc}\right)^5 dt = -\frac{Gm^2}{2R^2}\, dR \implies \int_0^t \frac{64G^3}{5}\frac{m^3}{c^5}\, dt = \int_d^{2r} -R^3\, dR$$

Solving this leads us to:

$$\frac{64G^3m^3}{5c^5}t = \frac{d^4 - r^4}{4} \implies t = \frac{5c^5(d^4 - 16r^4)}{256G^3m^3}$$

Plugging in the numbers gives:

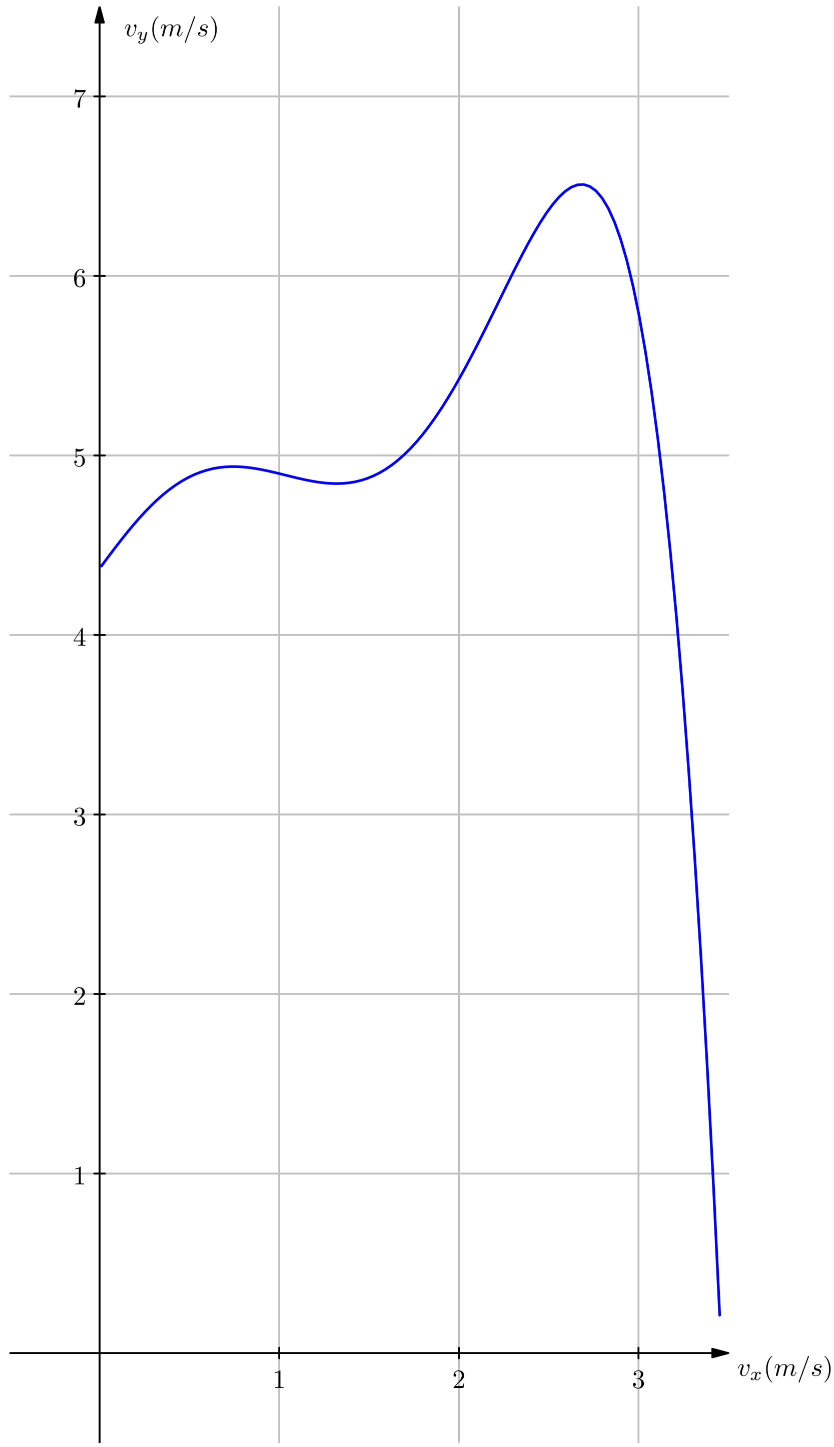
$$\boxed{t = 590 \text{ sec}}$$

Note that we can also assume that $d^4 \gg (2r)^4$ which will simplify calculations, and not introduce any noticeable error.

Pr 23. Too Bored

The graph provided plots the y-component of the velocity against the x-component of the velocity of a kiddie roller coaster at an amusement park for a certain duration of time. The ride takes place entirely in a two dimensional plane.

Some students made a remark that at one time, the acceleration was perpendicular to the velocity. Using this graph, what is the minimum x-velocity the ride could be travelling at for this to be true? Round to the nearest integer and answer in meters per second. The diagram is drawn to scale, and you may print this page out and make measurements.



Solution: The solution revolves around the idea that when the acceleration is perpendicular to the velocity, the work done is 0, and thus, the instantaneous rate of change of the magnitude of velocity $v_x^2 + v_y^2$ is 0. Thus, at such points, when the vertical velocity is plotted against the horizontal velocity, the curve will be tangent to a circle centered at the origin because $v_y^2 + v_x^2$ is nonchanging at that instant.

This is equivalent to stating that the line from the origin to the curve is perpendicular to the curve. Drawing such lines to the curve, the first time this occurs is at $v_x = \boxed{1 \text{ m/s}}$.

The following information applies to the next two problems: In the cosmic galaxy, the Sun is a main-sequence star, generating its energy mainly by nuclear fusion of hydrogen nuclei into helium. In its core, the Sun fuses hydrogen to produce deuterium (2H) and tritium (3H), then makes about 600 million metric tons of helium (4He) per second. Of course, there are also some relatively smaller portions of fission reactions in the Sun's core, e.g. a nuclear fission reaction with Uranium-235 (235U). The Fusion reaction:

$$^2H + {}^3H \to {}^4He + n + \text{Released Energy}$$

The Fission reaction:

$$^{235}U + n + (\text{Initial Energy}) \to {}^{144}Ba + {}^{90}Kr + 2n + \text{Released Energy}$$

Isotope Mass (at rest)

Isotope Names	Mass (at rest) (u)
Deuterium (^{2}H)	2.0141
Tritium (^{3}H)	3.0160
Helium (^{4}He)	4.0026
Neutron (n)	1.0087
Uranium-235 (^{235}U)	235.1180
Barium-144 (^{144}Ba)	143.8812
Krypton-90 (^{90}Kr)	89.9471

Pr 24. You are my Sunshine 1

Calculate the kinetic energy (in MeV) released by the products in one fusion reaction.

Pr 25. You are my Sunshine 2

Calculate the energy produced in the core of the Sun per second from helium fusion. Answer in Joules.

Solution:
(24) Let the kinetic energy released be

$$KE_{released} = -\Delta mc^2$$

Let the mass of helium be m_h, deuterium be m_d, tritium m_t, and mass of neutron m_n Therefore.

$$-\Delta m = m_d + m_t - m_h - m_n = 3.0160 + 2.0141 - 4.0026 - 1.0087 = 0.0188 \text{ u}$$

which gives

$$KE_{released} = (0.0188 \text{ u}) \cdot \left(931.494 \frac{\text{MeV}}{\text{u}}\right) = \boxed{17.51 \text{ MeV}}$$

(25) We perform the following conversions:

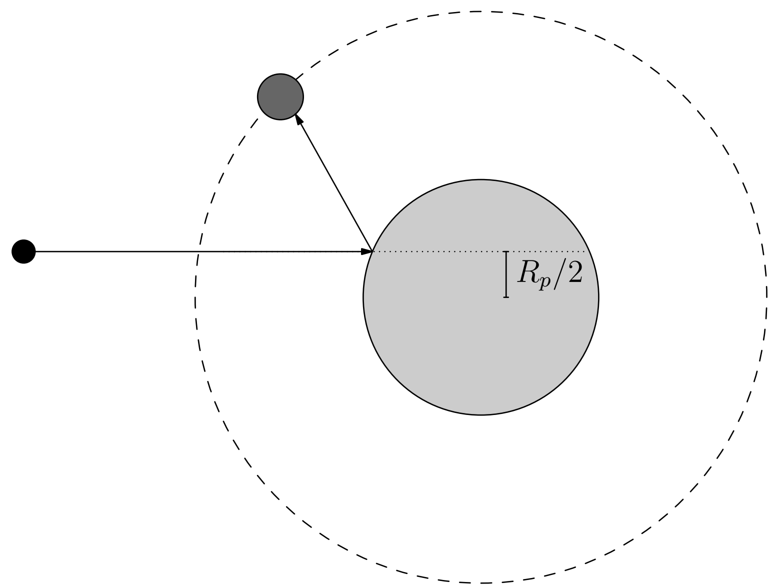
$$600 \cdot 10^6 \text{ tons He} = 6 \cdot 10^{14} \text{ g He}$$

$$\frac{6 \cdot 10^{14} \text{ g He}}{4.0026 \frac{\text{g He}}{\text{mol He}}} = 1.499 \cdot 10^{14} \text{ mol He}$$

$$1.499 \cdot 10^{14} \text{ mol He} \cdot \frac{6.02 \cdot 10^{23} \text{ molecules He}}{1 \text{ mol He}} \cdot \frac{17.51 \text{ MeV}}{1 \text{ molecule He}} \cdot \frac{1.6 \cdot 10^{-13} \text{ J}}{1 \text{ MeV}} = \boxed{2.528 \cdot 10^{26} \text{ J}}$$

Pr 26. Be Reflected It Must

While exploring outer space, Darth Vader comes upon a purely reflective spherical planet with radius $R_p = 40,000$ m and mass $M_p = 8.128 \times 10^{24}$ kg. Around the planet is a strange moon of orbital radius $R_s = 6,400,000$ m ($R_s \gg R_p$) and mass $M_s = 9.346 \times 10^{19}$ kg($M_s \ll M_p$). The moon can be modelled as a blackbody and absorbs light perfectly. Darth Vader is in the same plane that the planet orbits in. Startled, Darth Vader shoots a laser with constant intensity and power $P_0 = 2 \times 10^{32}$ W at the reflective planet and hits the planet a distance of $\frac{R_p}{2}$ away from the line from him to the center of the planet. Upon hitting the reflective planet, the light from the laser is plane polarized. The angle of the planet's polarizer is always the same as the angle of reflection. After reflectance, the laser lands a direct hit on the insulator planet. Darth Vader locks the laser in on the planet until it moves right in front of him, when he turns the laser off. Determine the energy absorbed by the satellite. Assume the reflective planet remains stationary and that the reflective planet is a perfect polarizer of light.



Solution: The original angle of reflectance can be found with some optical geometry to be $\theta_0 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. By Malus' Law, when the laser hits the planet with an angle θ, the final power after reflection is $P_0 \cos^2(\theta)$. Also, note that the satellite has to be at an angle 2θ due to the law of reflection. Therefore, by Kepler's Third law, the time for the satellite to reach $\theta = 0$ is

$$t = 2\theta\sqrt{\frac{r^3}{GM_p}} \implies dt = 2\sqrt{\frac{r^3}{GM_p}}d\theta.$$

Thus, the total power absorbed by the satellite is

$$\int_0^{\theta_0} P_0 \cos^2(\theta)dt = 2P_0\sqrt{\frac{r^3}{GM_p}}\int_0^{\theta_0} \cos^2(\theta)d\theta = P_0\sqrt{\frac{r^3}{GM_p}}(\theta_0 + \sin(\theta_0)\cos(\theta_0)) = \boxed{1.33 \cdot 10^{35}\text{J}}.$$

Pr 27. Braking Up

A particle of rest mass m moving at a speed $v = 0.7c$ decomposes into two photons which fly off at a separated angle θ. What is the minimum value of the angle of separation assuming that the two photons have equal wavelength. (Answer in degrees)

Solution: Conservation of momentum and energy gives:

$$p_m = 2E_\gamma \cos(\theta/2)$$

$$E_m = 2E_\gamma$$

Relativity demands that:

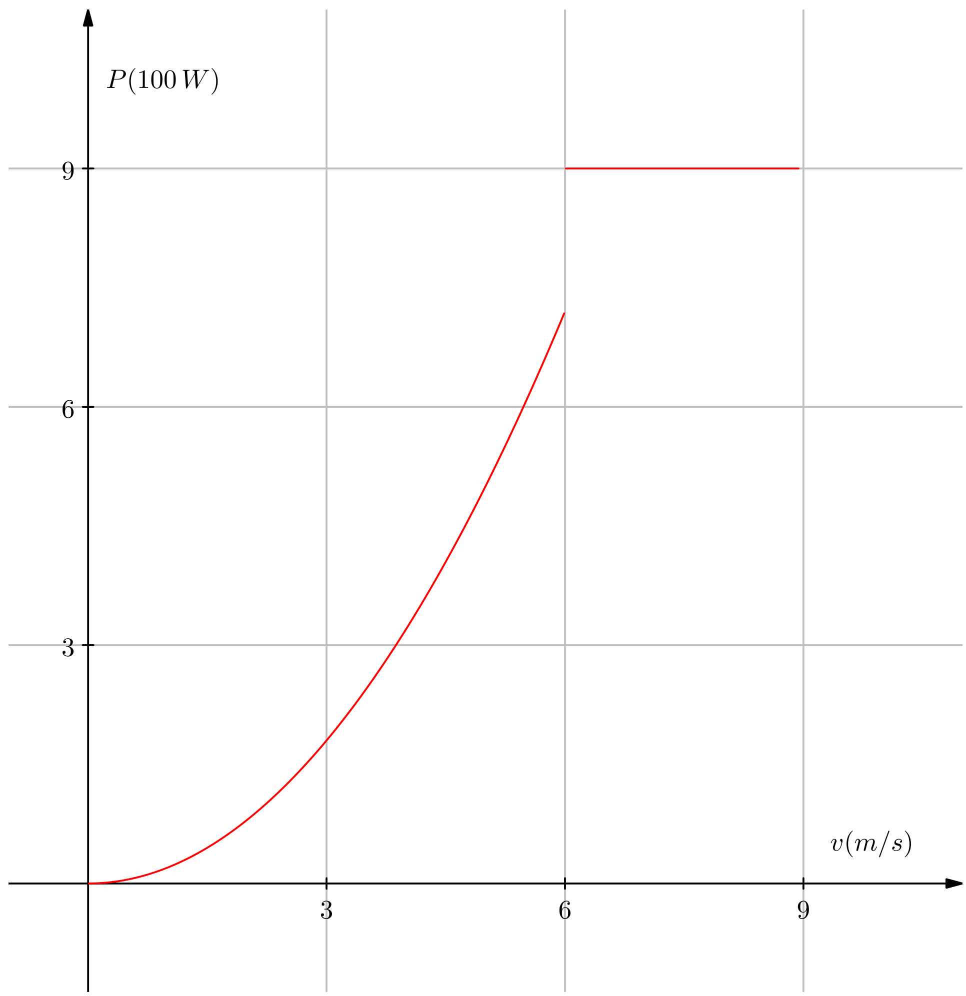
$$E_m^2 = m^2 + p_m^2$$

Solving this system of three equations gives:

$$\begin{aligned}
4E_\gamma^2 &= m^2 + p_m^2 \\
\frac{\gamma^2 m^2 v^2}{\cos^2(\theta/2)} &= m^2 + \gamma^2 m^2 v^2 \\
\gamma^2 v^2 \left(-1 + \frac{1}{\cos^2(\theta/2)}\right) &= 1 \\
\frac{1}{\cos^2(\theta/2)} &= \frac{1}{\gamma^2 v^2} + 1 \\
\cos(\theta/2) &= v \\
\theta &= \boxed{91.1^\circ}
\end{aligned}$$

Pr 28. With Great Power

Mario is racing with Wario on Moo Moo Meadows when a goomba, ready to avenge all of his friends' deaths, came and hijacked Mario's kart. A graph representing the motion of Mario at any instant is shown below. The velocity acquired by Mario is shown on the x-axis, and the net power of his movement is shown on the y-axis. When Mario's velocity is 6 m/s, he eats a mushroom which gives him a super boost.



You may need to make measurements. Feel free to print this picture out as the diagram is drawn to scale. Find the total distance from Mario runs from when his velocity is 0 m/s to when his velocity just reaches 9 m/s given that Mario's mass is $m = 89$ kg. Answer in meters and round to one significant digit.

Solution: Our first goal is to find an expression for the power curve $P(v)$. To do this, let us select a few points on the curve. The easiest point to pick is $(0, 0)$ since it is fixed at the origin. The next two easiest points to pick are those that are on the lines $x = 3$ and $x = 6$ which are given as approximately $(3, 1.8)$ and $(6, 7.2)$. Note that y-axis is in units of 100 W so in reality these two points are given as $(3, 180)$ and $(6, 720)$. This curve is resemblant of a quadratic in the form of $y = k_1x^2$ and upon solving for k_1 we find that the curve is given as $P = 20v^2$. Secondly, the next line remains constant with respect to time as a line $P = 900$. Therefore, we can write a piecewise function for power defined by

$$P(v) = \begin{cases} 20v^2 & \text{if } v \geq 0, \quad \text{and} \quad v < 6 \\ 900 & \text{if } v \geq 6 \end{cases}.$$

We need to find the relationship between power, velocity, and displacement of Mario. Consider dividing the displacement into tiny rectangular pieces with width Δt such that

$$s = \sum_{i \in \mathbb{N}} \Delta s_i = \sum_i v_i(t) \cdot \Delta t.$$

We want to the displacement to be expressed in terms without Δt. This means that we have to find a relationship for Δt. Note that

$$\Delta t = \Delta v \cdot \frac{\Delta t}{\Delta v} = \Delta v \cdot \frac{1}{\Delta v / \Delta t} = \frac{\Delta v}{a}.$$

Therefore, we now know the displacement to be expressed as

$$s = \sum_{i \to 0} v_i(t) \frac{\Delta v}{a} = \int \frac{v}{a(v)} \, dv.$$

To find an expression for $a(v)$ in terms of power, we note that

$$P(v) = F(v) \cdot v \implies a(v) = \frac{P(v)}{vm}$$

which means that upon substituting,

$$s = \int \frac{v^2 m}{P(v)} \, dv \implies s = \int_0^6 \frac{89v^2}{20v^2} dv + \int_6^9 \frac{89v^2}{900} dv = 43.61 \approx \boxed{40 \text{ m}}.$$

Pr 29. I'm a little teacup

At an amusement park, there is a ride with three "teacups" that are circular with identical dimensions. Three friends, Ethan, Rishab, and Kushal, all pick a teacup and sit at the edge. Each teacup rotates about its own axis clockwise at an angular speed $\omega = 1$ rad/s and can also move linearly at the same time.

The teacup Ethan is sitting on (as always) is malfunctional and can only rotate about its own axis. Rishab's teacup is moving linearly at a constant velocity 2 m/s [N] and Kushal's teacup is also moving linearly at a constant velocity of 4 m/s [N 60° E]. All three teacups are rotating as described above. Interestingly, they observe that at some point, all three of them are moving at the same velocity. What is the radius of each teacup?

Note: [N 60° E] means 60° clockwise from north e.g. 60° east of north.

Solution: We can plot the motion on a $v_y - v_z$ graph instead of carrying out calculations. We have three points at locations $(0,0)$, $(0,2)$, and $(2\sqrt{3},2)$ which represent the velocity of the center of mass of the teacups. The velocity that they are moving at can be traced as a circle with radius $r\omega$, centered at these points.

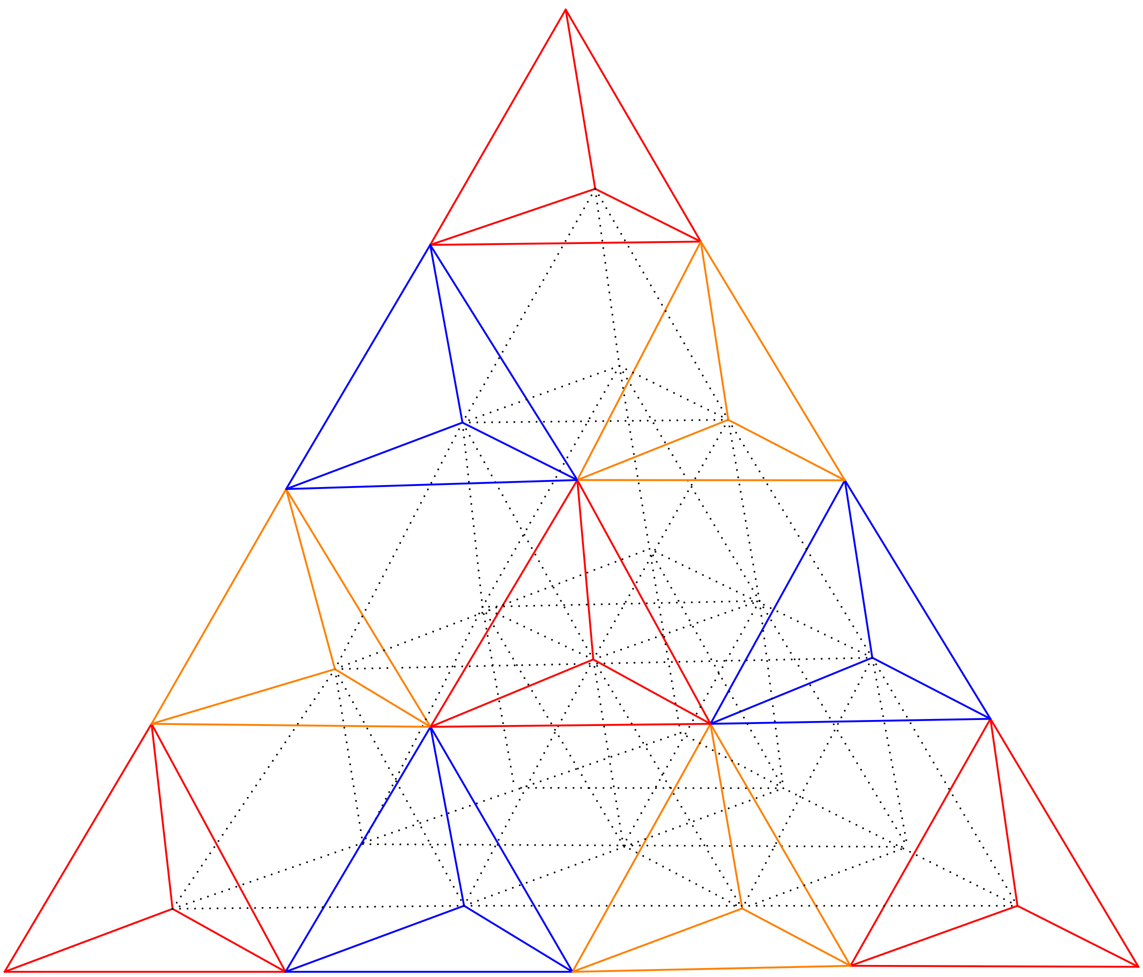
The problem now becomes, at what value r will the three circles intersect. Drawing a diagram, or carrying out trigonometric calculations gives $r = \boxed{2 \text{ m}}$.

Pr 30. Tetrahedron Resistance

An engineer has access to a tetrahedron building block with side length $\ell = 10$ cm. The body is made of a thermal insulator but the edges are wrapped with a thin copper wiring with cross sectional area $S = 2$ cm^2. The thermal conductivity of copper is 385.0 W/(m K). He stacks these tetrahedrons (all facing the same direction) to form a large lattice such that the copper wires are all in contact. In the diagram, only the front row of a small section is coloured. Assume that the lattice formed is infinitely large.

At some location in the tetrahedral building block, the temperature difference between two adjacent points is 1°C. What is the heat flow across these two points? Answer in Watts.

Note: Two adjacent points refer to two adjacent points on the tetrahedron.



Solution: There are many ways to solve this problem. We first identify that this is exactly the same as an infinite lattice resistor problem. To solve these, we can imagine injecting a current at a node and seeing how this current spreads out. However, a faster approach is by applying **Foster's Theorem** on this lattice.

The resistance of a single wire is:

$$R = \frac{\ell}{kS} = 1.299 \text{ W/K}$$

Foster's theorem tells us that

$$ER = V - 1$$

where V is number of vertices and E is edges. Taking the limit as $E, V \to \infty$, we get: $E = 6V$ (since each vertex is connected to 12 edges, but each edge is shared by two vertices). Therefore:

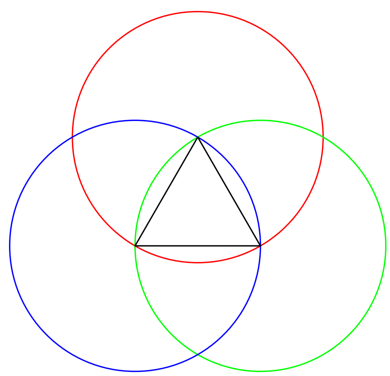
$$R_{\text{eff}} = \frac{1}{6}R = 0.2165 \text{ W/K}$$

From Fourier's Law, we have:

$$\dot{Q} = \frac{\Delta T}{R_{\text{eff}}} = \boxed{4.62 \text{ W}}.$$

Pr 31. AIME

Three unit circles, each with radius 1 meter, lie in the same plane such that the center of each circle is one intersection point between the two other circles, as shown below. Mass is uniformly distributed among all area enclosed by at least one circle. The mass of the region enclosed by the triangle shown above is 1 kg. Let x be the moment of inertia of the area enclosed by all three circles (intersection, *not* union) about the axis perpendicular to the page and through the center of mass of the triangle. Then, x can be expressed as $\frac{a\pi - b\sqrt{c}}{d\sqrt{e}}$ kg m^2, where a, b, c, d, e are integers such that $\gcd(a, b, d) = 1$ and both c and e are squarefree. Compute $a + b + c + d + e$.



Solution: Define point O as the point in the plane that the axis of rotation passes through. Since moments of inertia simply add about a given axis, we can calculate the moments of inertia of the three "sectors" whose union forms the given area and subtract twice the moment of inertia of the triangle, so our answer will be $3I_{s,O} - 2I_{t,O}$.
Claim: The center of mass of a sector is $\frac{2}{\pi}$ away from the vertex of the sector along its axis of symmetry.
Proof: We can divide the sector into arbitrarily small sectors that can be approximated as isosceles triangles. It's well known that the center of mass of one such isosceles triangle is $\frac{2}{3}$ of the way from the central vertex to the base. Therefore, the center of mass of the sector is the center of mass of the arc with central angle $\frac{\pi}{3}$ and same center with radius $\frac{2}{3}$ contained within the sector. Since the center of mass has to lie on the axis of symmetry, we set that as the x axis with the vertex of the sector being $x = 0$. Then, the x-coordinate of a point on the arc whose corresponding radius makes an angle of θ with the axis of symmetry is $\frac{2}{3}\cos(\theta)$. We can integrate this over all possible angles ($-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$) and then divide by the range ($\frac{\pi}{3}$) to get the average x-coordinate, or the center of mass.

$$\frac{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2}{3}\cos(\theta)\mathrm{d}\theta}{\frac{\pi}{3}}$$

$$\frac{2}{\pi}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(\theta)\mathrm{d}\theta$$

$$\frac{2}{\pi}\left(\sin\left(\frac{\pi}{6}\right) - \sin\left(-\frac{\pi}{6}\right)\right)$$

$$\frac{2}{\pi}$$

This concludes the proof. □
Now define point X as the vertex of a sector and point M as the center of mass of that sector. According to the parallel axis theorem,

$$I_{s,X} = I_{s,M} + m_s \left(\frac{2}{\pi}\right)^2$$

. It's well known that $I_{s,X} = \frac{1}{2}m_s r^2 = \frac{m_s}{2}$, and so

$$I_{s,M} = \frac{m_s}{2} - \frac{4m_s}{\pi^2} = m_s \left(\frac{\pi^2 - 8}{2\pi^2}\right)$$

It's also well known that O is on the line of symmetry and a distance of $\frac{1}{\sqrt{3}}$ away from X, and so $MX = \frac{2}{\pi} - \frac{1}{\sqrt{3}}$. Therefore,

$$I_{s,O} = I_{s,M} + m_s \left(\frac{2}{\pi} - \frac{1}{\sqrt{3}}\right)^2 = m_s \left(\frac{5\pi - 8\sqrt{3}}{6\pi}\right)$$

It's well known that, since O is the center of mass of the triangle,

$$I_{t,O} = \frac{1}{12}$$

Now we just need to calculate m_s. Since the mass of the triangle is 1 kg, this is equivalent to finding the ratio of the area of a sector to the area of a triangle. Through geometry, this is found to be $\frac{2\pi}{3\sqrt{3}}$. Finally, we get our answer to be

$$\left(\frac{2\pi}{\sqrt{3}}\right)\left(\frac{5\pi - 8\sqrt{3}}{6\pi}\right) - \frac{1}{6} = \left(\frac{10\pi - 17\sqrt{3}}{6\sqrt{3}}\right)$$

and $a + b + c + d + e = 10 + 17 + 3 + 6 + 3 = \boxed{039}$

Pr 32. Global Warming

Life on Earth would not exist as we know it without the atmosphere. There are many reasons for this, but one of which is temperature. Let's explore how the atmosphere affects the temperature on Earth. Assume that all thermal energy striking the earth uniformly and ideally distributes itself across the Earth's surface.

- Assume that the Earth is a perfect black body with no atmospheric effects. Let the equilibrium temperature of Earth be T_0. (The sun outputs around 3.846×10^{26} W, and is 1.496×10^{8} km away.)
- Now assume the Earth's atmosphere is isothermal. The short wavelengths from the sun are nearly unaffected and pass straight through the atmosphere. However, they mostly convert into heat when they strike the ground. This generates longer wavelengths that do interact with the atmosphere. Assume that the albedo of the ground is 0.3 and e, the emissivity and absorptivity of the atmosphere, is 0.8. Let the equilibrium average temperature of the planet now be T_1.

What is the percentage increase from T_0 to T_1?

Note: The emissivity is the degree to which an object can emit longer wavelengths (infrared) and the absorptivity is the degree to which an object can absorb energy. Specifically, the emissivity is the ratio between the energy emitted by an object and the energy emitted by a perfect black body at the same temperature. On the other hand, the absorptivity is the ratio of the amount of energy absorbed to the amount of incident energy.

Solution: Let us solve this problem in the case of the Earth being a graybody first and then substitute values for when it is a blackbody. The portion of energy that reaches the Earth is given by the ratio between the cross-sectional area of the satellite and the area of an imaginary sphere centered around the sun with a radius of L. Thus, the incoming radiation is multiplied by a factor of $\gamma = (R/2L)^2$. The energy from the sun that the surface absorbs is $\gamma(1-\alpha)E$, where E is the energy output of the sun. Here $\gamma = 1/4$ as the sphere encompassing will be 4 times the area of its intercept.
We can now write two systems of equations at the atmosphere and the ground of the Earth. At the top of the atmosphere, we require equilibrium meaning that zero net radiation leaves the atmosphere or:

$$-\frac{1}{4}S_0(1-\alpha) + \varepsilon\sigma T_a^4 + (1-\varepsilon)\sigma T_s^4 = 0.$$

Similarly, at the ground, we write another equilibrium equation of:

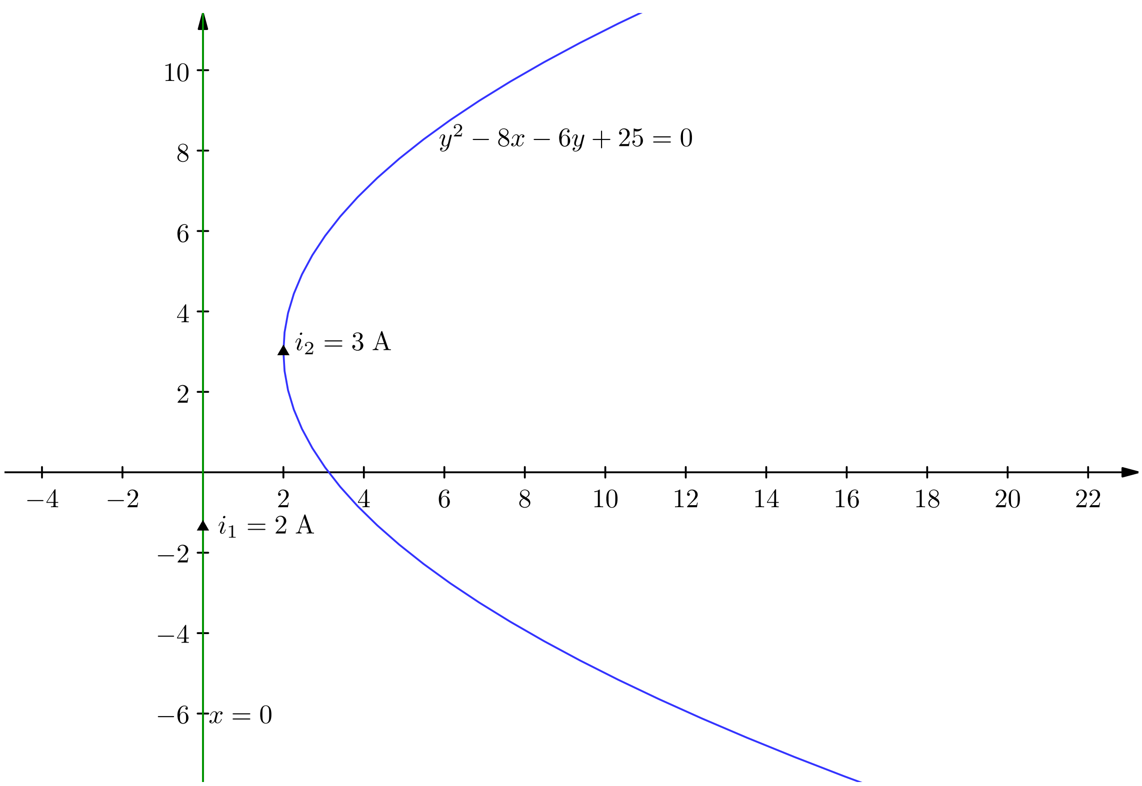
$$\frac{1}{4}S_0(1-\alpha) + \varepsilon\sigma T_a^4 - \sigma T_s^4 = 0.$$

Thus, solving the ground equilibrium equation yields us $T_a = 2^{-1/4}T_s$ and plugging back into the atmosphere equilibrium equation tells us:

$$\frac{1}{4}S_0(1-\alpha) = \left(1 - \frac{\varepsilon}{2}\right)\sigma T_s^4 \implies T_s = \boxed{289.601 \text{ K}}.$$

Pr 33. Flattening the Curve

Two infinitely long current carrying wires carry constant current $i_1 = 2$ A and $i_2 = 3$ A as shown in the diagram. The equations of the wire curvatures are $y^2 - 8x - 6y + 25 = 0$ and $x = 0$. Find the magnitude of force (in Newtons) acting on one of the wires due to the other.



Note: The current-carrying wires are rigidly fixed. The units for distances on the graph should be taken in metres.

Solution: The magnetic field from the wire is given by $B = \frac{\mu_0 i_1}{2\pi x}$. Let θ be the direction of a component of force from the vertical. It is then seen that

$$dF = Bi_2 d\ell \implies dF_x = Bi_2 d\ell \sin\theta = Bi_2 dy.$$

We only consider the force in the x-direction which means that

$$F_x = \int_{\infty}^{\infty} dF_x = \frac{\mu_0 i_1 i_2}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{x}.$$

Solving the equation in terms of x and then plugging in gives us

$$F_x = \frac{8\mu_0 i_1 i_2}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{y^2 - 6y + 25} = \frac{8\mu_0 i_1 i_2}{2\pi}\cdot\frac{\pi}{4} = \mu_0 i_1 i_2 = \boxed{7.5398\cdot 10^{-6}\text{ N}}.$$

Pr 34. Hiking in Mountains

Mountains have two sides: windward and leeward. The windward side faces the wind and typically receives warm, moist air, often from an ocean. As wind hits a mountain, it is forced upward and begins to move towards the leeward side. During social distancing, Rishab decides to cross a mountain from the windward side to the leeward side of the mountain. What he finds is that the air around him has warmed when he is on the leeward side of the mountain.

Let us investigate this effect. Consider the warm, moist air mass colliding with the mountain and moving upwards on the mountain. Disregard heat exchange with the air mass and the mountain. Let the humidity of the air on the windward side correspond to a partial vapor pressure 0.5 kPa at 100.2 kPa and have a molar mass of $\mu_a = 28$ g/mole. The air predominantly consists of diatomic molecules of oxygen and nitrogen. Assume the mountain to be very high which means that at the very top of the mountain, all of the moisture in the air condenses and falls as precipitation. Let the precipitation have a heat of vaporization $L = 2.4\cdot 10^6$ J/kg and molar mass $\mu_p = 18.01$ g/mole. Calculate the total change in temperature from the windward side to the leeward side in degrees Celsius.

Solution: We use the first law of thermodynamics to solve this problem. For diatomic molecules, the internal energy per mole is given by $\frac{5}{2}RT$. If the molar mass of the air is μ_a, then we have that the change in internal energy of the air is given by

$$\Delta U = \frac{5}{2}\frac{M}{\mu_a}R\Delta T.$$

We also note that the total work performed by the gas is

$$W = P_2V_2 - P_1V_1$$

since the process is adiabatic, we can use the ideal gas equation $PV = \nu RT = (M/\mu)RT$ to express the total work as

$$W = \frac{M}{\mu_a}R\Delta T.$$

The heat that is taken away during condensation at the top of the mountain is given by $Q = L\Delta m$ where Δm is the total mass of the precipitation. According to the ideal gas law, we have that

$$PV_1 = \frac{\Delta m}{\mu_p}RT_1, \quad P_1V_1 = \frac{M}{\mu_a}RT_1$$

recombining these equations and equating them gives us

$$\frac{\Delta m}{\mu_p P}RT_1 = \frac{M}{\mu_a P_1}RT_1$$

$$\Delta m = M\frac{\mu_p P}{\mu_a P_1}.$$

Therefore,

$$Q = LM\frac{\mu_a P}{\mu_p P_1}.$$

We finally can now use the first law of thermodynamics

$$Q = \Delta U + W \implies LM\frac{\mu_p P}{\mu_a P_1} = \frac{5}{2}\frac{M}{\mu_a}R\Delta T + \frac{M}{\mu_a}R\Delta T.$$

We then simplify this equation to get

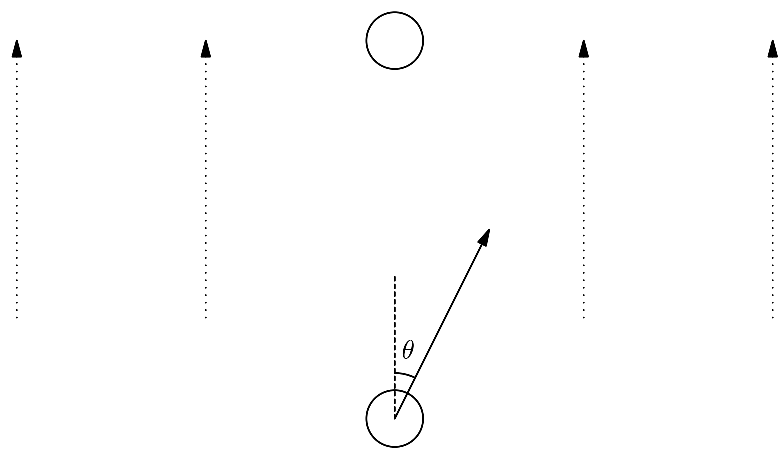
$$LM\frac{\mu_p P}{\mu_a P_1} = \frac{7}{2}\frac{M}{\mu_a}R\Delta T$$
$$\Delta T = \frac{2}{7}\frac{L\mu_p P}{RP_1} = \boxed{7.41 \text{ K}}$$

A simpler approach could be to assume that the number of moles of water vapour in the atmosphere is equal to number of moles of water condensed. Then, the mass of precipitated water is $\mu_p n \frac{P}{P_1}$, where n is the number of moles of air. Thus,

$$\mu_p n\frac{P}{P_1}L = \frac{7}{2}nR\Delta T.$$

Pr 35. Me and my Crush

Two electrons are in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ where $E_0 = 10^{-11}$ N/C. One electron is at the origin, and another is 10 m above the first electron. The electron at the origin is moving at $u = 10$ m/s at an angle of 30° from the line connecting the electrons at $t = 0$, while the other electron is at rest at $t = 0$. Find the minimum distance between the electrons. You may neglect relativistic effects.



Solution: Let $\ell = 10$ m. First, switch into the reference frame accelerating at $-\frac{Eq}{m}\hat{z}$. In this frame, the electrons are not affected by the electric field. Now, switch into the center of mass reference frame from here. In this frame, we have both conservation of angular momentum and conservation of energy. Both electrons in this frame are moving at $\frac{u}{2}$ initially at an angle of $\theta = 30°$. At the smallest distance, both electrons will be moving perpendicular to the line connecting them. Suppose that they both move with speed v and are a distance r from the center of mass. By conservation of angular momentum,

$$2m \cdot \frac{u}{2} \cdot \frac{\ell}{2}\sin\theta = 2mvr$$

$$vr = \frac{u\ell}{4}\sin\theta.$$

Now, by conservation of energy,

$$mv^2 + \frac{ke^2}{2r} = \frac{1}{4}mu^2 + \frac{ke^2}{\ell}.$$

Now, we just solve this system of equations to determine the value of r. Substituting $v = \frac{ul}{4r}\sin\theta$ into the conservation of energy equation, we can solve the ensuing quadratic to find:

$$r = \frac{\frac{ke^2}{2} + \sqrt{\left(\frac{ke^2}{2}\right)^2 + \left(mu^2 + \frac{4ke^2}{\ell}\right)\left(\frac{mu^2\ell^2}{16}\sin^2(\theta)\right)}}{\frac{1}{2}mu^2 + \frac{2ke^2}{\ell}}.$$

Finally, remembering that the distance between the electrons is actually $2r$, we obtain $2r = \boxed{6.84 \text{ m}}$ as the final answer.

Pr 36. Can't or can

Consider a long uniform conducting cylinder. First, we divide the cylinder into thirds and remove the middle third. Then, we perform the same steps on the remaining two cylinders. Again, we perform the same steps on the remaining four cylinders and continuing until there are 2048 cylinders.

We then connect the terminals of the cylinder to a battery and measure the effective capacitance to be C_1. If we continue to remove cylinders, the capacitance will reach an asymptotic value of C_0. What is C_1/C_0?

You may assume each cylindrical disk to be wide enough to be considered as an infinite plate, such that the radius R of the cylinders is much larger than the d between any successive cylinders.

Note: The diagram is not to scale.

Solution: The capacitance is proportional to $C \propto \frac{1}{d}$, where d is the distance between successive parallel plates. When we add capacitor plates in series, their effective capacitance will be:

$$C \propto \left(\frac{1}{1/d_1} + \frac{1}{1/d_2} + \cdots\right)^{-1} = \frac{1}{d_1 + d_2 + \cdots} \implies C \propto \frac{1}{d_{\text{total}}}$$

Therefore, this essentially becomes a math problem: What is the total length of the spacing in between? Between successive 'cuts', the length of each cylinder is cut down by 1/3, but the number of gaps double. Therefore, the spacing grows by a factor of 2/3 each time. For $n = 2^1$, the spacing starts off as 1/3. For $n = 2^{10}$, the spacing becomes:

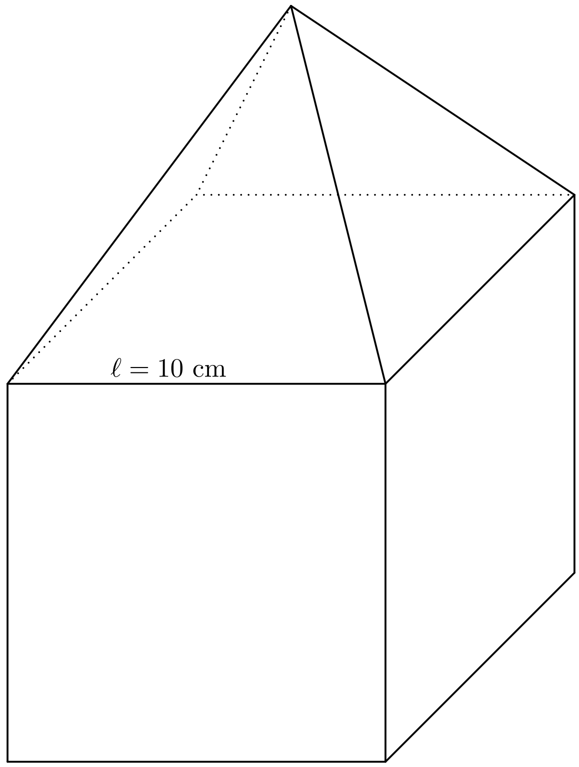
$$\frac{1}{C_{\text{eff}}} \propto d = \frac{1}{3}\left(\frac{1-(2/3)^{10}}{1-2/3}\right)L = 0.983L$$

for $n \to \infty$, it is clear the total spacing will converge to L. Therefore:

$$C_1/C_0 = \boxed{1.017}$$

Pr 37. Mom Trust the Physics!

A square based pyramid (that is symmetrical) is standing on top of a cube with side length $\ell = 10$ cm such that their square faces perfectly line up. The cube is initially standing still on flat ground and both objects have the same uniform density. The coefficient of friction between every surface is the same value of $\mu = 0.3$. The cube is then given an initial speed v in some direction parallel to the floor. What is the maximum possible value of v such that the base of the pyramid will always remain parallel to the top of the cube? Answer in meters per second.



Solution: Let x be the relative displacement of the two objects. Then:

$$v_i^2 = 2ax$$

where the relative acceleration is:

$$a = \frac{2h}{3\ell} g\mu$$

work The acceleration of pyramid is $g\mu$ and the acceleration of cube is:

$$\rho\ell^3 a = \rho\left(\ell^3 + \frac{\ell^2 h}{3}\right) g\mu + \rho\frac{\ell^2 h}{3} g\mu \implies a = \frac{3\ell + 2h}{3\ell} g\mu$$

Therefore, the relative acceleration is:

$$a = \frac{6\ell + 2h}{3\ell} g\mu$$

We want to direct the motion of the cube diagonally (such that horizontal sides of the cube form a 45 degree angle with the displacement). Initially, we may think that we need to let $x = \frac{\sqrt{2}}{2}\ell$ but it can start tipping before that. Moving into a non-inertial reference frame for the pyramid, we see that the effective gravity needs to point towards the back corner of the cube, so it needs to satisfy the criteria

$$\frac{h_{\text{cm}}}{\frac{\sqrt{2}}{2}\ell - x} = \frac{mg}{mg\mu} \implies x = \frac{\sqrt{2}}{2}\ell - \mu\frac{1}{4}h$$

Here I used the fact that the center of mass was 1/4 of the way up. Substituting, we get:

$$v = \sqrt{2\left(\frac{6\ell + 2h}{3\ell} g\mu\right)\left(\frac{\sqrt{2}}{2} l - \frac{\mu}{4} h\right)} = \sqrt{2g\mu\left(2 + \frac{2h}{3\ell}\right)\left(\frac{\sqrt{2}}{2}\ell - \frac{\mu}{4} h\right)}$$

The maximum v occurs at

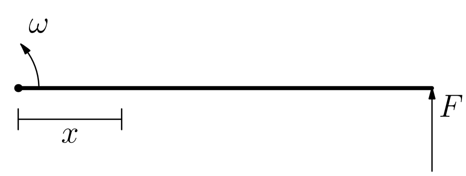
$$h/\ell = \frac{-3 + \frac{2\sqrt{2}}{\mu}}{2}$$

giving a maximum v of $v_{\max} = \boxed{1.07 \text{ m/s}}$.

Pr 38. FBI Open Up!

During quarantine, the FBI has been monitoring a young physicists suspicious activities. After compiling weeks worth of evidence, the FBI finally has had enough and searches his room.

The room's door is opened with a high angular velocity about its hinge. Over a very short period of time, its angular velocity increases to $\omega = 8.56$ rad/s due to the force applied at the end opposite from the hinge. For simplicity, treat the door as a uniform thin rod of length $L = 1.00$ m and mass $M = 9.50$ kg. The hinge (pivot) is located at one end of the rod. Ignore gravity. At what distance from the hinge of the door is the door most likely to break? Assume that the door will break at where the bending moment is largest. (Answer in metres.)



Solution: Let N be the force from the pivot and F be the applied force at the end. Let α be the angular acceleration. Writing the torque equation and Newton's 2nd law for the whole door, we get:

$$F \cdot L = \frac{1}{3} M L^2 \alpha$$

$$N + F = \frac{1}{2} M L \alpha$$

Solving, we get $F = \frac{1}{3} M L \alpha$ and $N = \frac{1}{6} M L \alpha$. Now, we consider the part of the door with length x attached to the pivot. The rest of the door applies a torque τ and shear force f on our system. (There is also tension force). Let $\lambda = \frac{M}{L}$. We can write the torque equation and Newton's 2nd law for our system:

$$\tau + fx = \frac{1}{3} \lambda x^3 \alpha$$

$$N + f = \lambda x \cdot \frac{x}{2} \alpha$$

Solving, we get

$$\tau = \frac{1}{6} \lambda x \alpha (L^2 - x^2)$$

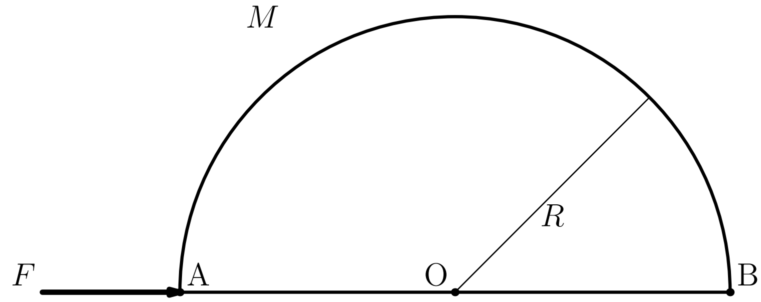
and

$$f = \frac{1}{6} \lambda \alpha (3x^2 - L^2)$$

We maximize τ (which is equivalent to maximizing bending moment) to get $x = \frac{L}{\sqrt{3}} = \boxed{0.577 \text{ m}}$

Pr 39. Pappu's Half Disk

A solid half-disc of mass $m = 1$ kg in the shape of a semi-circle of radius $R = 1$ m is kept at rest on a smooth horizontal table. QiLin starts applying a constant force of magnitude $F = 10$ N at point A as shown, parallel to its straight edge. What is the initial linear acceleration of point B? (Answer in m/s^2)



Note: the diagram above is a *top down view*.

Solution: Let C denote the location of the centre of mass of the disc. It is well known that $OC = \frac{4R}{3\pi}$. Note that the initial angular velocity of the disc about its centre of mass is 0, and the linear acceleration is simply

$$a_{\text{CM}} = \frac{F}{m}\hat{i}$$

Now we use the $\tau = \vec{r} \times \vec{F} = I_{\text{CM}}\alpha_{\text{CM}}$ about the centre of mass of the disc

$$\left(\frac{4R}{3\pi}\right) F = I_{\text{CM}}\alpha_{\text{CM}}$$

To compute the moment of inertia of the disc about its centre of mass, we use Steiner's theorem:

$$I_{\text{CM}} = \frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2$$

so the $\tau = I\alpha$ equation becomes

$$\left(\frac{4R}{3\pi}\right) F = \left[\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2\right]\alpha_{\text{CM}} \Rightarrow \alpha_{\text{CM}} = \frac{\frac{4FR}{3\pi}}{MR^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)}$$

Using kinematics equation for rotational motion, we have

$$\vec{a}_B = \vec{a}_{\text{CM}} + \vec{\alpha}_{\text{CM}} \times \vec{\text{CB}}$$

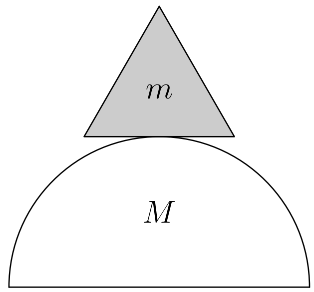
Substituting the values of $\vec{\text{CB}} = R\hat{i} - \frac{4R}{3\pi}\hat{j}$ and $\alpha_{\vec{\text{CM}}} = \frac{\frac{4FR}{3\pi}}{MR^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)}\hat{k}$ we get

$$a_B = \frac{F}{m}\left[\left(1 + \frac{1}{-1 + \frac{9\pi^2}{32}}\right)\hat{i} + \frac{2}{3\pi\left(\frac{1}{2} + \frac{16}{9\pi^2}\right)}\hat{j}\right] = \boxed{15.9395 \text{ m/s}^2}$$

and we are done. □

Pr 40. Don't Fall

A regular tetrahedron of mass $m = 1$ g and unknown side length is balancing on top of a hemisphere of mass $M = 100$ kg and radius $R = 100$ m. The hemisphere is placed on a flat surface such that it is at its lowest potential. For a certain value of the length of the regular tetrahedron, the oscillations become unstable. What is this side length of the tetrahedron?



Solution: For stable equilibrium the height resultant from a slight displacement must be greater than the original height of the center of mass of the cube. A small displacement from the original position can be modeled as an x displacement of

$$Rd\theta,$$

which raises the height by

$$Rd\theta \sin(d\theta)$$

added to the height

$$s \cos(d\theta)$$

This must be greater than s, where s is the distance from the center of mass to the point of contact to the sphere. Approximating to the second degree of θ using Taylor series, we solve the inequality and get that $s = R$. The altitude of the tetrahedron is therefore $4s = 400$

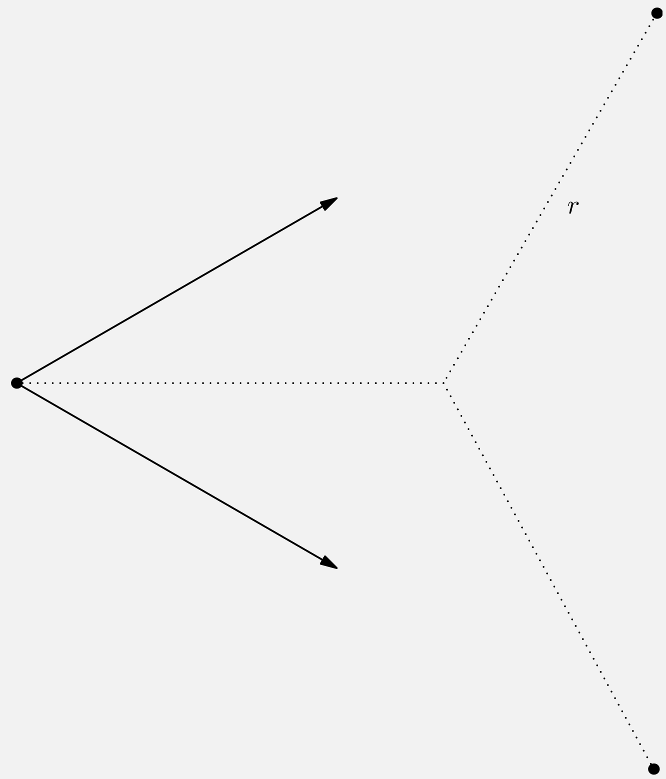
$$l^2 - (l\sqrt{3}/3)^2 = 400^2$$

Therefore, $l = \sqrt{400^2 \cdot 3/2} = \boxed{490 \text{ m}}$

Pr 41. Heartbreak

A planet has a radius of 10 km and a uniform density of 5 g/cm^3. A powerful bomb detonates at the center of the planet, releasing 8.93×10^{17} J of energy, causing the planet to separate into three large sections each with equal masses. You may model each section as a perfect sphere of radius r'. The initial and final distances between the centers of any two given sections is $2r'$. How long does it take for the three sections to collide again?

Solution: Due to conservation of momentum, the three masses must form an equilateral triangle at all times. Let us determine the force as a function of r, the distance between each mass and the center.



The vector sum of the net force on any individual mass is

$$F = \frac{Gm^2}{d^2}\sqrt{2 - 2\cos 120^\circ} = \frac{\sqrt{3}Gm^2}{d^2}$$

where d is the distance between the mass and the center.

$$d^2 = 3r^2$$

The net force is thus

$$F = \frac{Gm^2}{\sqrt{3}r^2}$$

The system behaves as if there was a stationary mass $m' = m/\sqrt{3}$ at the center, simplifying the problem greatly into a restricted two body system. Next, we need to figure out the height of the apoapsis. This can be done via conservation of energy.

$$E_{\text{binding,initial}} + E = 3E_{\text{binding,final}} - \frac{Gm^2}{\sqrt{3}\ell} \implies \ell = -\frac{Gm^2}{\sqrt{3}\left(-\frac{3GM^2}{5R} + 8.93 \cdot 10^{17} - \left(-3\frac{3Gm^2}{5r_f}\right)\right)} = 101,000 \text{ m}$$

If you have a stationary mass M at the center. The time it takes for an object to fall into it is:

$$T = \pi\sqrt{\frac{\ell^3}{8GM}}$$

our time will be double this, and the mass in the center will be $M = m/\sqrt{3}$. So plugging in numbers gives:

$$t = 2\pi\sqrt{\frac{l^3}{8G\left(\frac{m}{\sqrt{3}}\right)}} = 138,000$$

If we take into account a nonzero radius so final separation is $10/\cos(30°)$ km, then the answer should be:

$$-2\int_{l}^{\frac{10000}{\cos\left(\frac{\pi}{6}\right)}}\left(\sqrt{\frac{xl}{2G\left(\frac{m}{\sqrt{3}}\right)(l-x)}}\right)dx = \boxed{136,000 \text{ s}}$$

Note if you set the upper bound to zero, you get the same answer as before. Both these answers will be accepted.

Pr 42. Sandwiched!

A point charge $+q$ is placed a distance a away from an infinitely large conducting plate. The force of the electrostatic interaction is F_0. Then, an identical conducting plate is placed a distance $3a$ from the charge, parallel to the first one such that the charge is "sandwiched in." The new electrostatic force the particle feels is F'. What is F'/F_0? Round to the nearest hundredths.

Solution: We solve this via the method of image charges. Let us first reflect the charge $+q$ across the closest wall. The force of this interaction will be:

$$F_0 = \frac{q^2}{4\pi\epsilon_0 a^2}\frac{1}{2^2}$$

and this will be the only image charge we need to place if there were only one conducting plane. Since there is another conducting plane, another charge will be reflected to a distance $a+4a$ past the other conducting plane, and thus will be $a+4a+3a=8a$ away from the original charge. All these reflections cause a force that points in the same direction, which we will label as the positive + direction. Therefore:

$$F_+ = \frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{2^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{16^2}\frac{1}{18^2}+\frac{1}{24^2}+\cdots\right)$$

Now let us look at what happens if we originally reflect the charge $+q$ across the other wall. Repeating the steps above, we see that through subsequent reflections, each force will point in the negative $-$ direction. Therefore:

$$F_- = \frac{-q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{14^2}+\frac{1}{16^2}+\frac{1}{22^2}+\frac{1}{24^2}+\cdots\right)$$

The net force is a result of the superposition of these two forces, giving us:

$$F' = \frac{q^2}{4\pi\epsilon_0 a^2}\Big(\frac{1}{2^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{16^2}\frac{1}{18^2}+\frac{1}{24^2}+\cdots -\frac{1}{6^2}-\frac{1}{8^2}-\frac{1}{14^2}-\frac{1}{16^2}-\frac{1}{22^2}-\frac{1}{24^2}-\cdots\Big)$$

Even terms can be cancelled out to give:

$$F' = \frac{q^2}{4\pi\epsilon_0 a^2}\left(\frac{1}{2^2}-\frac{1}{6^2}+\frac{1}{10^2}-\frac{1}{14^2}+\frac{1}{18^2}-\frac{1}{22^2}+\cdots\right)$$
$$= \frac{q^2}{4\pi\epsilon_0 a^2}\frac{1}{4}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots\right)$$

You may recognize the infinite series inside the parentheses to be Catalan's constant $G \approx 0.916$. Alternatively, you can use a calculator and evaluate the first seven terms to get a rough answer (but will still be correct since we asked for it to be rounded). Therefore:

$$F'/F = \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\cdots = G \approx \boxed{0.916}$$

The following information applies to the next three problems. Jerry spots a truckload of his favourite golden yellow Swiss cheese being transported on a cart moving at a constant velocity $v_0 = 5$ m/s $\hat{i}$ along the x-axis, which is initially placed at $(0, 0)$. Jerry, driven by desire immediately starts pursuing the cheese-truck in such a way that his velocity vector always points towards the cheese-truck; however, Jerry is smart and knows that he must maintain a constant distance $\ell = 10$ m from the truck to avoid being caught by anyone, no matter what. Note that Jerry starts at coordinates $(0, \ell)$.

Pr 43. Tom and Jerry 1

Let the magnitude of velocity (in m/s) and acceleration (in m/s^2) of Jerry at the moment when the (acute) angle between the two velocity vectors is $\theta = 60°$ be α and β respectively. Compute $\alpha^2 + \beta^2$.

Pr 44. Tom and Jerry 2

At a certain instant during Jerry's motion, when his distance from the x-axis is 2 m, let his distance from the y-axis be ξ (in metres), and let his speed at $t = 1$ second be ψ m/s. Compute $\xi^2 + \psi^2$.

Pr 45. Tom and Jerry 3

Tom spots Jerry's footprints in the mud after Jerry has already travelled a distance $\ell = 10$ m towards the cheese truck. He starts moving at a constant speed of 5 m/s (except for a very large acceleration at the start, a result of his dislike for Jerry) along Jerry's trail. Alas, as is destined, he will never be able to catch Jerry. After a long period of time, what will be the separation between them? (in meters) Assume that Tom and Jerry have the energy to maintain their velocities for a very long period of time. Tom starts chasing Jerry from the same place Jerry started running towards the cheese truck.

Solution:
(43) If the distance between Jerry and the cheese truck is constant, then Jerry moves in circle of radius ℓ in the reference frame of the cheese truck. There is no radial component of Jerry's velocity in this reference frame, so we must have $\alpha = v_0 \cos\theta = \frac{5}{2}$. In this case, the tangential velocity is $v_0 \sin\theta$. Furthermore, the radial acceleration in this frame is given by the centripetal acceleration which is $\frac{(v_0 \sin\theta)^2}{\ell} = \frac{v_0^2 \sin^2\theta}{\ell}$. The tangential acceleration is

$$\frac{d}{dt}(v_0 \sin\theta) = v_0 \cos\theta \cdot \frac{d\theta}{dt} = v_0 \cos\theta \cdot \frac{-v_0 \sin\theta}{\ell} = -\frac{v_0^2 \sin\theta \cos\theta}{\ell}.$$

The vector sum of these accelerations has magnitude

$$\beta = \sqrt{\left(\frac{v_0^2 \sin^2\theta}{\ell}\right)^2 + \left(\frac{v_0^2 \sin\theta \cos\theta}{\ell}\right)^2} = \frac{v_0^2 \sin\theta}{\ell} = \frac{5\sqrt{3}}{4}.$$

The final answer is $\alpha^2 + \beta^2 = 10.9375$.

(44) As in the previous part, we can work in the reference frame of the cheese truck. As in the previous problem, we know that

$$\frac{d\theta}{dt} = -\frac{v_0 \sin\theta}{\ell}.$$

We can solve this differential equation more explicitly:

$$\frac{d\theta}{\sin\theta} = -\frac{v_0}{\ell} dt$$

$$\int_{\frac{\pi}{2}}^{\theta} \frac{d\theta'}{\sin\theta'} = -\frac{v_0}{\ell} \int_0^t dt'.$$

$$\ln|\cot\theta + \csc\theta| = \frac{v_0}{\ell} t$$

From the value of t, we can solve for θ and find the speed from $\psi = v_0 \cos\theta = 2.3106$ (Note a quick way to find θ is to use $\cot\frac{\theta}{2} = \cot\theta + \csc\theta$). Now, the distance from the x-axis is given by $\ell \sin\theta$, so we can easily find θ and substitute in to our equation to find the time. At this time, the cheese truck has moved a distance $v_0 t$, but Jerry is a horizontal distance $\ell\cos\theta$ behind the truck, so the distance to the y-axis is $\xi = v_0 t - \ell\cos\theta = 13.1264$. Finally, $\xi^2 + \psi^2 = 177.6$.

(45) First, we will calculate the change in the horizontal distance between Tom and the cheese truck from the time Tom starts moving. When Jerry was moving along this path, in a small time dt, the angle of Jerry's motion changes by $d\theta = -\frac{\ell}{v_0 \sin\theta} d\theta$ from the results in the previous problem. For the Tom's motion, the speed is faster by a factor of $\frac{1}{\cos\theta}$, so in time dt for the Tom, we have $dt = -\frac{\ell}{v_0}\cot\theta d\theta$. Now, the cheese truck continues to the right at speed v_0, while Tom has a horizontal velocity $v_0\cos\theta$. Thus, the total change in horizontal distance between Tom and the cheese truck is

$$\int_0^\infty (v_0 - v_0\cos\theta)dt = \int_{\frac{\pi}{2}}^0 (v_0 - v_0\cos\theta)\left(-\frac{\ell}{v_0}\cot\theta d\theta\right) = \ell\int_0^{\frac{\pi}{2}}(\cot\theta - \cos\theta\cot\theta)d\theta = \ell(1-\ln 2).$$

The initial horizontal distance between Tom and the cheese truck can be found with $v_0 t$ where t is the time at which Jerry has traveled a distance ℓ. The arc length of Jerry's path is

$$\int (v_0\cos\theta)dt = \int_{\frac{\pi}{2}}^0 (-\ell\cot\theta)d\theta = -\ell\ln|\sin\theta| = \ell.$$

Thus, we find $\sin\theta = \frac{1}{e}$ and $\cos\theta = \frac{\sqrt{e^2-1}}{e}$. From our equation above, distance the cheese truck travels in this time is

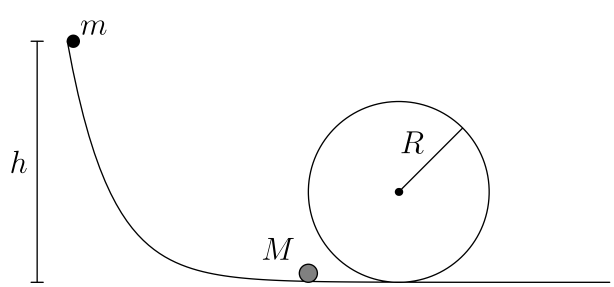
$$v_0 t = \ell\ln|\cot\theta + \csc\theta| = \ell\ln(e + \sqrt{e^2-1}).$$

The distance between Tom and the cheese truck then approaches $\ell(\ln(e+\sqrt{e^2-1}) + 1 - \ln 2)$ after a long time. Since Jerry lags the truck by a distance ℓ, the distance between Tom and Jerry approaches

$$\ell(\ln(e+\sqrt{e^2-1}) - \ln 2) = \boxed{9.64 \text{ m}}.$$

Pr 46. Ghoster Coaster

A frictionless track contains a loop of radius $R = 0.5$ m. Situated on top of the track lies a small ball of mass $m = 2$ kg at a height h. It is then dropped and collides with another ball of mass $M = 5$ kg.



The coefficient of restitution for this collision is given as $e = \frac{1}{2}$. Now consider a different alternative. Now let the circular loop have a uniform coefficient of friction $\mu = 0.6$, while the rest of the path is still frictionless. Assume that the balls can once again collide with a restitution coefficient of $e = \frac{1}{2}$. Considering the balls to be point masses, find the minimum h such that the ball of mass M would be able to move all the way around the loop. Both balls can be considered as point masses.

Solution: Let the angle formed by M at any moment of time be angle θ with the negative y-axis. The normal force experienced by M is just

$$N = Mg\cos\theta + M\frac{v(\theta)^2}{R}$$

by balancing the radial forces at this moment. Now, applying the work energy theorem, we have

$$\int -\mu\left[Mg\cos\theta + M\frac{v(\theta)^2}{R}\right]R\mathrm{d}\theta = \frac{1}{2}Mv(\theta)^2 - \frac{1}{2}M{v_0}^2 + MgR(1-\cos\theta)$$

$$\Rightarrow -\mu\left[Mg\cos\theta + M\frac{v(\theta)^2}{R}\right]R = \frac{M}{2}\frac{\mathrm{d}(v(\theta)^2)}{\mathrm{d}\theta} + MgR\sin\theta$$

Rearranging, we have

$$\frac{\mathrm{d}(v(\theta)^2))}{\mathrm{d}\theta} + 2\mu v(\theta)^2 = -2gR(\sin\theta + \mu\cos\theta)$$

Let $v^2(\theta) = y$. Thus we have a first order linear ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} + P(\theta)y = Q(\theta)$$

This is easily solvable using the integrating factor $e^{\int P(\theta)\mathrm{d}\theta}$. Here the integrating factor is

$$e^{\int 2\mu\mathrm{d}\theta} = e^{2\mu\theta}$$

So multiplying by the integrating factor, we get

$$\int \mathrm{d}(e^{2\mu\theta}y) = \int -2gR(\sin\theta + \mu\cos\theta)e^{2\mu\theta}\mathrm{d}\theta$$

$$\Rightarrow y = \frac{\int -2gR(\sin\theta + \mu\cos\theta)e^{2\mu\theta}\mathrm{d}\theta}{e^{2\mu\theta}}$$

Now we use the well known integrals

$$\int e^{ax}\sin x\ \mathrm{d}x = \frac{e^{ax}}{1+a^2}(a\sin x - \cos x)$$

$$\int e^{ax}\cos x\ \mathrm{d}x = \frac{e^{ax}}{1+a^2}(a\cos x + \sin x)$$

(These integrals can be computed using integration by parts.) Thus, plugging and chugging these integration formulas into our expression for y and integrating from $\theta = 0$ to $\theta = \phi$, we have upon solving

$$v^2(\phi) - {v_0}^2 = \frac{-2gR}{1+4\mu^2}\left[(3\mu\sin\phi + (2\mu^2-1)\cos\phi - (2\mu^2-1)e^{-2\mu\phi}\right]$$

where v_0 is the velocity at $\phi = 0$. Solving gives us the velocity as a function of angle covered

$$v(\phi) = \sqrt{{v_0}^2 - \frac{2gR}{1+4\mu^2}\left[(3\mu\sin\phi + (2\mu^2-1)\cos\phi - (2\mu^2-1)e^{-2\mu\phi}\right]}$$

But to cover a complete circle, at the top most point

$$N = mg - \frac{mv^2(\pi)}{R} \geq 0 \Rightarrow v(\pi) \leq \sqrt{gR}$$

Thus

$$v_0 \leq \sqrt{gR\left[1 + \frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

From the previous expression,

$$v_0 = \frac{m(1+e)\sqrt{2gh}}{M+m} \geq \sqrt{gR\left[1+\frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

Hence

$$\boxed{h \geq \frac{R(M+m)^2}{2m^2(1+e)^2}\left[1+\frac{2(1-2\mu^2)}{1+4\mu^2}(1+e^{-2\mu\pi})\right]}$$

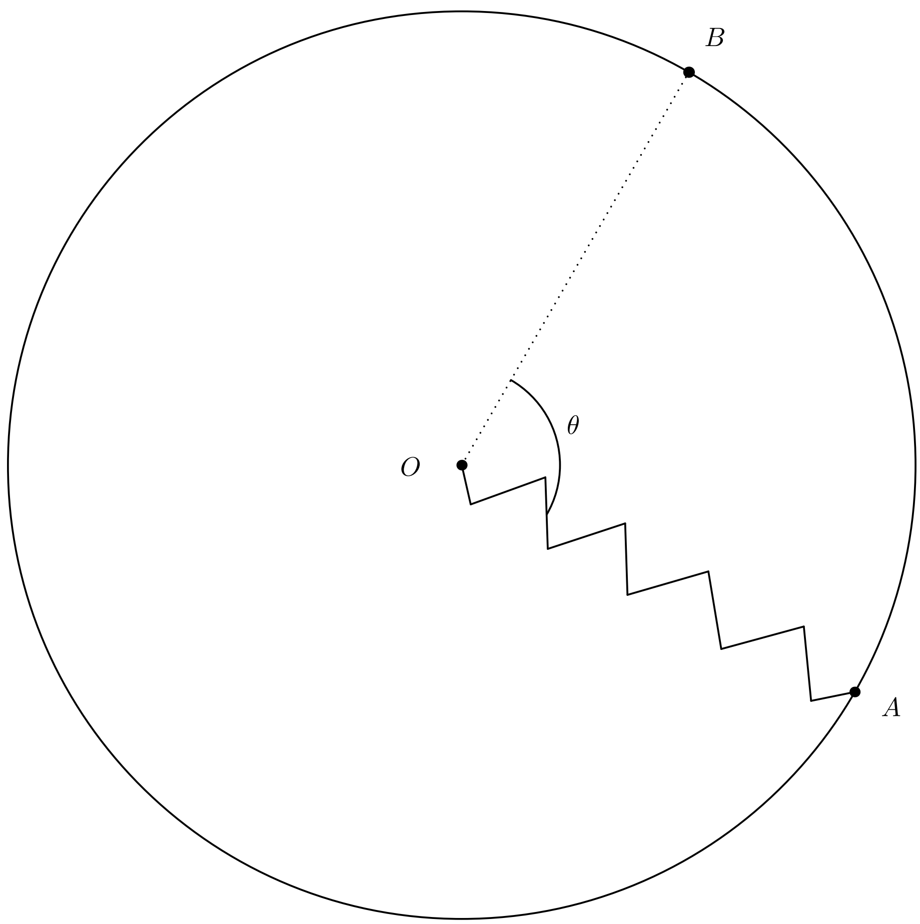
We get $h \geq \boxed{72.902 \text{ m}}$ and we are done.

Pr 47. Galactic Games

Two astronauts, Alice and Bob, are standing inside their cylindrical spaceship, which is rotating at an angular velocity ω clockwise around its axis in order to simulate the gravitational acceleration g on earth. The radius of the spaceship is R. For this problem, we will only consider motion in the plane perpendicular to the axis of the spaceship. Let point O be the center of the spaceship. Initially, an ideal zero-length spring has one end fixed at point O, while the other end is connected to a mass m at the "ground" of the spaceship, where the astronauts are standing (we will call this point A). From the astronauts' point of view, the mass remains motionless.

Next, Alice fixes one end of the spring at point A, and attaches the mass to the other end at point O. Bob starts at point A, and moves an angle θ counterclockwise to point B (such that AOB is an isosceles triangle). At time $t = 0$, the mass at point O is released. Given that the mass comes close enough for Bob to catch it, find the value of θ to the nearest tenth of a degree.

Assume that the only force acting on the mass is the spring's tension, and that the astronauts' heights are much less than R.



Solution: First, we must have $\omega^2 R = g$ and $k = m\omega^2$. Now, we step into the frame of point A, rotating around with the spaceship. We will thus have three fictitious forces: translational, centrifugal, and coriolis. Note that because centrifugal is $m\omega^2 r$, pointing away from A, it cancels with the spring force. Thus, the only forces left to consider are translational and coriolis.

The translational force points "down" with a constant magnitude of mg, like gravity. The coriolis force

points perpendicular to the velocity with magnitude $2m\omega v$. We recognize this setup is analogous to that of a charged particle moving in an E field and B field. It is well known that the mass will follow a cycloid shape. Writing the equation of the cycloid, and finding where the cycloid hits the circle (spaceship), we can find θ. Note that we have to use numerical methods.
Specifically, the cycloid can be parametrized as $\frac{R}{4}(\alpha - \sin\alpha, 1 - \cos\alpha)$, and we need to find where this intersects the circle $x^2 + y^2 = R^2$, so $(\alpha - \sin\alpha)^2 + (1 - \cos\alpha)^2 = 16$. Solving gives $\alpha = 3.307$, and since $\cot\theta = \frac{1-\cos\alpha}{\alpha - \sin\alpha}$, we have $\theta = \boxed{60.2^\circ}$.

Pr 48. Cramped Up

Consider an LC circuit with one inductor and one capacitor. The amplitude of the charge on the plates of the capacitor is $Q = 10$ C and the two plates are initially at a distance $d = 1$ cm away from each other. The plates are then slowly pushed together to a distance 0.5 cm from each other. Find the resultant amplitude of charge on the parallel plates of the capacitor after this process is completed. Note that the initial current in the circuit is zero and assume that the plates are grounded.

Solution: In slow steady periodic processes (when the time for the change in parameters τ is much less than the total systems frequency f), a quantity called the adiabatic invariant I is conserved[a]. The adiabatic invariant corresponds to the area of a closed contour in phase space (a graph with momentum p and position x as its axes). Note the we can electrostatically map this problem to a mechanics one as the charge corresponds to position, while the momentum would correspond to LI where I is the current and L is the inductance. Thus, in phase space, we have an elliptical contour corresponding to the equation: $\frac{Q^2}{2C} + \frac{(LI)^2}{2L} = C$ where C is a constant in the system. As the area under the curve is conserved, then it can be written that $\pi Q_0 L I_0 = \pi Q_f L I_f$. It is also easy to conserve energy such that $LI^2 = \frac{Q^2}{C}$ which tells us $I = \frac{Q}{\sqrt{LC}}$. As $C \propto 1/x$, we then can write the adiabatic invariant as xq^4 which tells us $\boxed{Q_f = \sqrt[4]{2}Q}$.

We can also solve this regularly by looking at the changes analytically. From Gauss's law, the electric field between the plates of the capacitators initially can be estimated as

$$E = \frac{Q}{2\varepsilon_0 A}$$

where A is the area of the plate. The plates of the capacitator is attracted to the other one with a force of

$$F = QE = \frac{Q^2}{2\varepsilon_0 A}.$$

The charges of the plates as a function of time can be approximated as

$$Q_c = \pm Q \sin(\omega t + \phi).$$

where $\omega = \frac{1}{\sqrt{LC}}$. Using this equation, we estimate the average force $\langle F \rangle$ applied on the plate after a period of oscillations to be

$$\langle F \rangle = \frac{\langle Q^2 \rangle}{2\varepsilon_0 A} = \frac{Q^2}{2\varepsilon_0 A}\left\langle \sin^2(\omega t + \phi) \right\rangle = \frac{Q^2}{2\varepsilon_0 A} \cdot \left(\frac{1}{2\pi} \int_0^{2\pi} \sin^2(x) dx \right) = \frac{Q^2}{4\varepsilon_0 A}$$

this means that after one period, the amount of work done to push the plates closer together is given by

$$W_F = \langle F \rangle \, dx = \frac{Q^2}{4\varepsilon_0 A} dx.$$

In this cycle, the amount of incremental work done by the LC circuit will be given by

$$dW_{\text{LC}} = \Delta(Fx) = \Delta\left(\frac{Q^2 x}{2\varepsilon_0 A} \right) = \frac{Qx}{\varepsilon_0 A} dQ + \frac{Q^2}{2\varepsilon_0 A} dx.$$

From conservation of energy, $W_F = W_{LC}$. Or in other words,

$$\frac{Q^2}{4\varepsilon_0 A}dx = \frac{Qx}{\varepsilon_0 A}dQ + \frac{Q^2}{2\varepsilon_0 A}dx$$

simplifying gives us

$$\frac{Qx}{\varepsilon_0 A}dQ = -\frac{Q^2}{4\varepsilon_0 A}dx$$
$$\frac{1}{4}\int \frac{dx}{x} = -\int \frac{dQ}{Q}$$
$$\frac{1}{4}\ln x + \ln Q = \text{const.}$$

We now find our adiabatic invariant to be

$$xQ^4 = \text{const.}$$

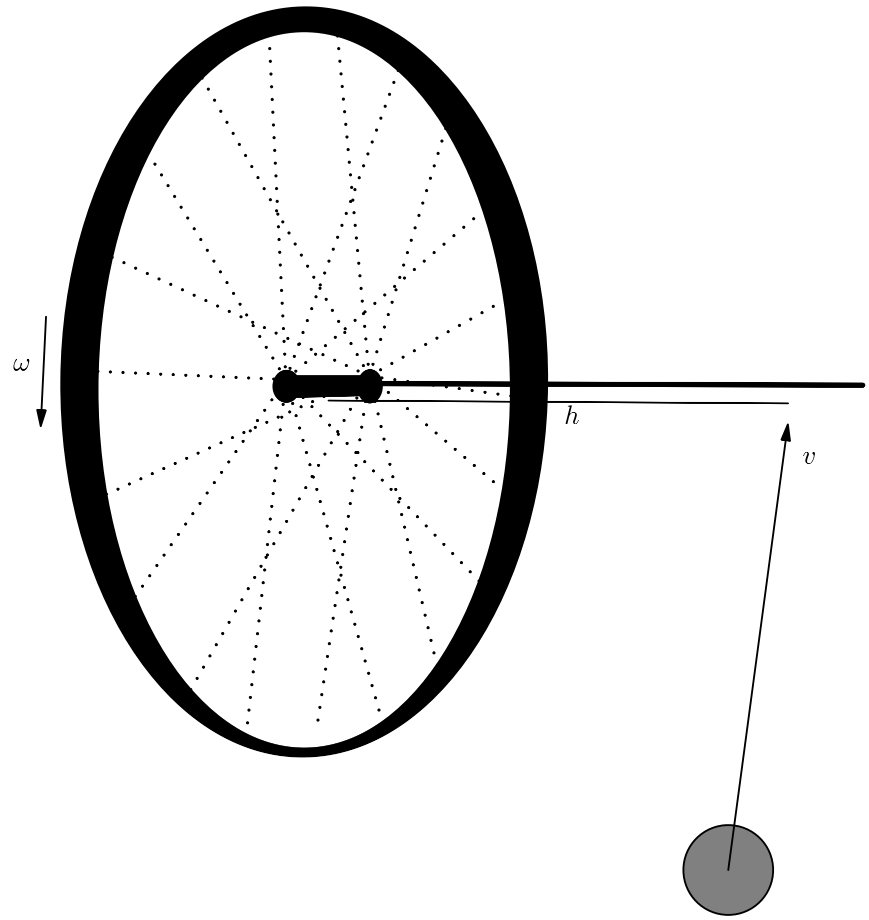
Substituting values into our equation, we find that

$$dQ_i^4 = \frac{d}{2}Q_f^4 \implies Q_f = \sqrt[4]{2}Q = \boxed{11.892 \text{ C}}$$

[a] This will not be proved in this solution, but the proof can be found in any good mechanics book.

Pr 49. I knew I should've stayed home today

A bicycle wheel of mass $M = 2.8$ kg and radius $R = 0.3$ m is spinning with angular velocity $\omega = 5$ rad/s around its axis in outer space, and its center is motionless. Assume that it has all of its mass uniformly concentrated on the rim. A long, massless axle is attached to its center, extending out along its axis. A ball of mass $m = 1.0$ kg moves at velocity $v = 2$ m/s parallel to the plane of the wheel and hits the axle at a distance $h = 0.5$ m from the center of the wheel. Assume that the collision is elastic and instantaneous, and that the ball's trajectory (before and after the collision) lies on a straight line.



Find the time it takes for the axle to return to its original orientation. Answer in seconds and round to three significant figures.

Solution: After the collision, let the wheel have speed v_1 and the ball have speed v_2. Conserving momentum, energy, and angular momentum gives:

$$mv = Mv_1 + mv_2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}MR^2\omega^2 + \frac{1}{2} \cdot \frac{1}{2}MR^2\omega_1^2$$

$$m(v - v_2)h = \frac{1}{2}MR^2\omega_1$$

where ω_1 is the angular velocity (after collision) of the wheel in the direction perp. to the axis and the velocity of the ball.

Solving for ω_1, we get

$$\omega_1 = \frac{4hmv}{m(R^2+2h^2)+MR^2}.$$

Now, we realize that the angular momentum of the wheel is given by $I_x\omega\hat{x} + I_y\omega_1\hat{y}$ where the wheel's axis is the x-axis and the y-axis is in the direction of ω_1. Since angular momentum is conserved, the wheel must precess about its angular momentum vector. Let $\hat{L}$ represent the direction of the angular momentum vector. To find the rate of precession, we can decompose the angular velocity vector $\omega\hat{x} + \omega_1\hat{y}$ into a $\hat{L}$ component and a $\hat{x}$ component. Since $I_x = 2I_y$, the $\hat{L}$ component is $\sqrt{(2\omega)^2+\omega_1^2}$, resulting in a precession period of

$$T = \frac{\pi}{\sqrt{\omega^2+\frac{\omega_1^2}{4}}} = \boxed{0.568s}$$

Pr 50. Fun with a String

A child attaches a small rock of mass $M = 0.800$ kg to one end of a uniform elastic string of mass $m = 0.100$ kg and natural length $L = 0.650$ m. He grabs the other end and swings the rock in uniform circular motion around his hand, with angular velocity $\omega = 6.30$ rad/s. Assume his hand is stationary, and that the elastic string behaves like a spring with spring constant $k = 40.0$ N/m. After that, at time $t = 0$, a small longitudinal perturbation starts from the child's hand, traveling towards the rock. At time $t = T_0$, the perturbation reaches the rock. How far was the perturbation from the child's hand at time $t = \frac{T_0}{2}$? Ignore gravity.

Solution: Let x be the distance from a point on the unstretched elastic string to the center of rotation (child's hand). Note that x varies from 0 to L. However, the string stretches, so let r be the distance from a point on the stretched string (in steady state) to the center of rotation. Let T be the tension in the string as a function of position. Let $\lambda = \frac{m}{L}$. Consider a portion of the string dx. We know that the portion as spring constant $k\frac{L}{dx}$ and it is stretched by $dr - dx$, so by Hooke's Law, we have $T = k\frac{L}{dx}(dr - dx) = kL(\frac{dr}{dx} - 1)$. Also, by applying Newton's Second Law on the portion, we get $dT = -\lambda dx \cdot \omega^2 r$, which implies $\frac{dT}{dx} = -\lambda\omega^2 r$. Combining the two equations, we obtain

$$T = -kL\left(\frac{1}{\lambda\omega^2}T'' + 1\right).$$

We know that

$$T'(x=0) = 0,$$

since $r = 0$ when $x = 0$. The general solution is

$$T = A\cos\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right) - kL,$$

for some constant A. Thus, we have

$$r = -\frac{1}{\lambda\omega^2}T' = \frac{A}{\omega\sqrt{km}}\sin\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right).$$

Also, we have that

$$T(x=L) = M\omega^2\int_0^L rxdx = M\omega^2\int_0^L\left(\frac{T}{kL}+1\right)x.$$

Plugging in our general solution, we can get $A\cos\left(\omega\sqrt{\frac{m}{k}}\right) - kL = M\omega^2 \cdot \frac{A}{kL}\frac{L}{\omega}\sqrt{\frac{k}{m}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)$. Solving for A, we obtain

$$A = \frac{kL}{\cos\left(\omega\sqrt{\frac{m}{k}}\right) - \frac{M\omega}{\sqrt{km}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)}.$$

We now introduce a claim:

Claim. The speed of a longitudinal wave on a spring with spring constant k, length L, and mass m is given by $v = L\sqrt{\frac{k}{m}}$

Proof. Let a spring with spring constant k and mass m be stretched to length L. The spring constant of a small portion dx of the spring is $k\frac{L}{dx}$, and the excess tension is $\delta T = k\frac{L}{dx}ds = kL\frac{ds}{dx}$, where s is the displacement from equilibrium. By Newton's second law on the portion, we get $dT = \frac{m}{L}dx \cdot \frac{d^2s}{dt^2}$, or $\frac{dT}{dx} = \frac{m}{L}\frac{d^2s}{dt^2}$. Thus, $L^2\frac{k}{m}\frac{d^2s}{dx^2} = \frac{d^2s}{dt^2}$, which we recognize as the wave equation with speed $v = L\sqrt{\frac{k}{m}}$ and the time it takes to traverse the spring is $\sqrt{\frac{m}{k}}$. □

Thus, we have

$$t = \int dt = \int_0^x \sqrt{\frac{\lambda dx}{k \cdot \frac{L}{dx}}} = \int_0^x \sqrt{\frac{\lambda}{kL}}dx = \sqrt{\frac{m}{k}}\frac{x}{L}.$$

Since we know $x = L$ when $t = T_0$, we have $x = \frac{L}{2}$ when $t = \frac{T_0}{2}$. Therefore, our answer is

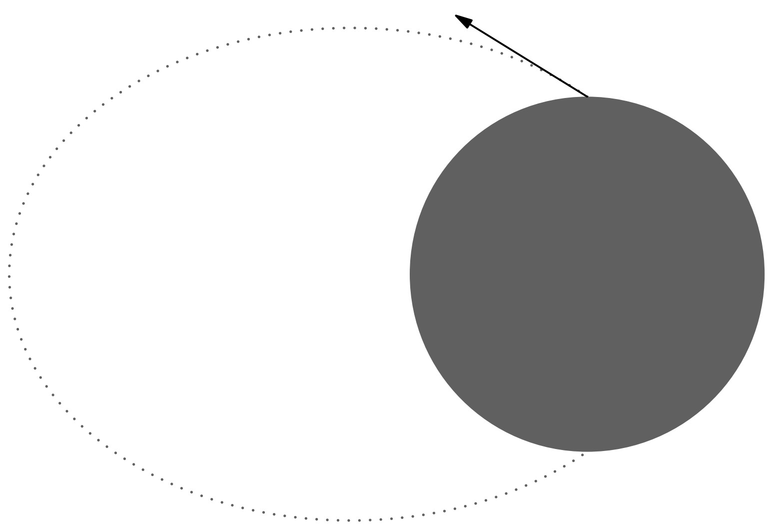
$$r = \frac{A}{\omega\sqrt{km}}\sin\left(\frac{\omega}{L}\sqrt{\frac{m}{k}}x\right) = \frac{1}{\omega}\sqrt{\frac{k}{m}}\frac{L\sin\left(\frac{1}{2}\omega\sqrt{\frac{m}{k}}\right)}{\cos\left(\omega\sqrt{\frac{m}{k}}\right) - \frac{M\omega}{\sqrt{km}}\sin\left(\omega\sqrt{\frac{m}{k}}\right)} = \boxed{1.903 \text{ m}}$$

Pr 51. When Rocket Scientists Play Catch

During the cold war, there was tension between the USSR and the U.S. But now, contrary to popular belief, American and Russian astronauts pass time by hanging out, enjoying the view from the moon, and even playing catch by launching projectiles at each other:

A projectile is launched with a speed $v_0 = 2200$ m/s from the North Pole to the South Pole of a moon with radius $r_0 = 1.7 \times 10^6$ m and $M = 7.4 \times 10^{22}$ kg.

How long does the flight take? Answer in seconds.



Solution: The projectile will follow an elliptical path. It is easiest to represent this path in terms of an ellipse:

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}$$

At $\theta = 90°$, the numerator becomes $a(1-e^2) = r_0$. We will now show that the angular momentum is given by:

$$L = m\sqrt{GMr_0}.$$

If the apoapsis is r_a and the speed of the projectile at apoapsis is v_a, then conservation of angular momentum and energy gives:

$$L = mv_f r_f$$

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f}$$

Solving these equations, we get:

$$r_a = GM \cdot \frac{-1 + \sqrt{\frac{v^2 r}{GM} - 1}}{2E}$$

where E is the energy per unit mass of the projectile. By setting $\theta = 180°$, we can write the location of the apoapsis to be

$$r_a = a(1+e) = 9274582m$$

Therefore, to determine the apoapsis a, we just need to determine e. The eccentricity is given by:

$$e = \sqrt{1 + \frac{2Er_0}{GM}} = 0.8167.$$

Plugging in numbers, we find the semi-major axis to be $a = 461,670$ m and the orbital period to be

$$T = 2\pi\sqrt{\frac{a^3}{GM}} = 32622 \text{ s}$$

However, we are only interested in the section of the orbit that occurs above the surface. If we are able to determine the area the center of the moon subtends with the curve inside the moon, then we can apply Kepler's second law to determine the orbital period.

This area can be determined via polar integration to be:

$$\frac{1}{2}\int_{\frac{5\pi}{2}}^{\frac{7\pi}{2}} \left(\frac{a\left(1-e_c^2\right)}{1-e_c\cos\theta}\right)^2 d\theta = 2.1633 \times 10^{12} \text{ m}^2$$

The fraction of area covered by the projectile while outside the moon will be:

$$f = \frac{\pi ab - 2.1633 \times 10^{12}}{\pi ab} = \frac{\pi a^2\sqrt{1-e^2} - 2.1633 \times 10^{12}}{\pi a^2\sqrt{1-e^2}} = 0.9542$$

where $b = c^2 - a^2$ and $c = e/a$ were used to simplify the expression. Applying Kepler's second law, the time of flight is then:

$$t = fT = \boxed{31128.8 \text{ s}}$$