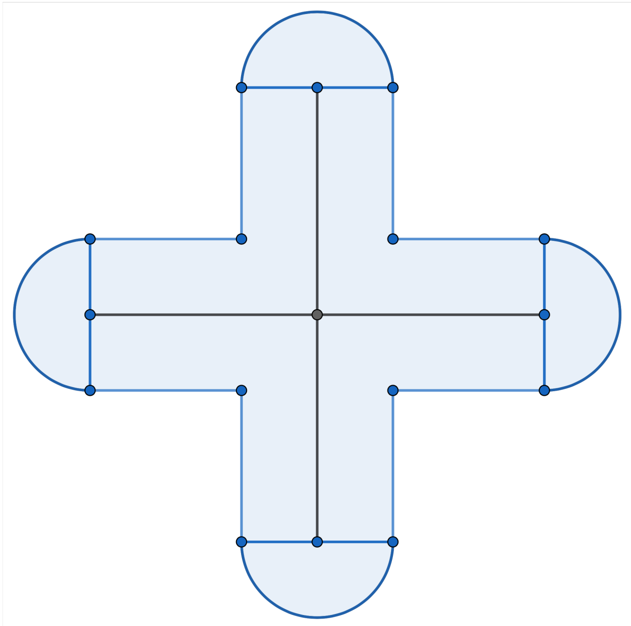
T1: Booster

Solution 1:
All point values should be multiplied by 3 (for a total of 30 points)

(a) (i) The shape of the chamber after time t is a circle with radius $r_0 + vt$. Thus the volume burned is $V = \pi(r_0+vt)^2 l - \pi r_0^2 l$, and $\dot{M} = \dot{V}\rho_s = \boxed{2\pi\rho_s l v(r_0+vt)}$

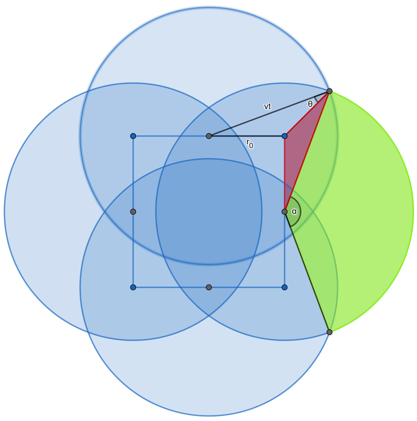
(ii) The volume of fuel that is burned after some time ($t \le r_0/v$) looks like this:



The total area of the semicircular endcaps is $2\pi v^2t^2$, and the area of the remaining cross-like figure is $4r_0^2 - 4(r_0-vt)^2$. So the total area is $A = 2\pi v^2t^2 + 4r_0^2 - 4(r_0-vt)^2$. The mass flow rate is $\boxed{\dot{M} = l\rho_s\dot{A} = 4l\rho_s((\pi-2)v^2t + 2r_0v)}$.

The intent of the problem was for $t \le r_0/v$, but that wasn't clearly specified. For fairness to everyone who spent time on it, we solve the $t > r_0/v$ case-https://www.overleaf.com/project/64cc0b0dcfb587f85f42416c below.

This case looks like the following:



In the figure, by Law of Sines, $\theta = \arcsin\frac{r_0}{\sqrt{2}vt}$. Then $\alpha = \pi - 2(\pi/4 - \arcsin\frac{r_0}{\sqrt{2}vt}) = \pi/2 + 2\arcsin\frac{r_0}{\sqrt{2}vt}$.

The total area of the circular sectors (colored green) is $2(vt)^2(\frac{\pi}{2} + 2\arcsin\frac{r_0}{\sqrt{2}vt})$. The total area of the triangles (colored red) is $4vtr_0\sin\left(\pi/4 - \arcsin\frac{r_0}{\sqrt{2}vt}\right) = 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1-\frac{r_0^2}{2v^2t^2}} -$

$\frac{r_0}{2vt}) = 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1-\frac{r_0^2}{2v^2t^2}} - \frac{r_0}{2vt})$. The area inside the square is constant, so differentiation will get rid of it.
We want:

$$\dot{M} = \rho_s l \frac{\mathrm{d}}{\mathrm{dt}}\left[2(vt)^2(\frac{\pi}{2} + 2\arcsin\frac{r_0}{\sqrt{2}vt}) + 4vtr_0(\frac{1}{\sqrt{2}}\sqrt{1-\frac{r_0^2}{2v^2t^2}} - \frac{r_0}{2vt})\right]$$
$$= \rho_s l(4v^2t(\pi/2 + 2\arcsin\frac{r_0}{\sqrt{2}vt}) - 4v^2t^2\frac{r_0}{vt^2\sqrt{2-\frac{r_0^2}{v^2t^2}}}$$
$$+ 4vr_0(\sqrt{2-\frac{r_0^2}{v^2t^2}} - \frac{r_0}{2vt}) + 4vtr_0(\frac{r_0^2}{2v^2t^3\sqrt{2-\frac{r_0^2}{v^2t^2}}} + \frac{r_0}{2vt^2}))$$

Grading Scheme

- **1 pts** for correct answer for (i)
- **2 pts** for correct answer for $t < r_0/v$ for (ii). 1 pt should be given if only a small mistake (e.g. neglecting the endcaps) was made in the figure.
- **1 pt** for something resembling the correct answer for $t > r_0/v$ for (ii)

(b) The gas is produced at a (volumetric) rate of $2\pi\frac{\rho_s}{\rho_g}lv(r_0+vt)$, and the area of the opening is $\pi(r_0+vt)^2$, so the velocity is

$$v_g = 2\frac{\rho_s}{\rho_g}\frac{lv}{r_0+vt}$$

Multiplying by the mass flow rate gives

$$\boxed{F = 4\pi\frac{\rho_s}{\rho_g}\rho_s l^2 v^2}$$

Which is time-independent.
Grading Scheme

- **2 pts** for correct final answer.

(c) The assumption is false. If the pressure and temperature stay constant, then by conservation of energy, the velocity of the exhaust gasses will be constant. Thus the thrust will be proportional to area and not constant in time, which contradicts part (b).
Grading Scheme

- **1 pts** for stating the assumption is false and a valid explanation.

(d) The speed of sound is $v = \sqrt{\frac{3RT}{M}}$ as only 3 of the degrees of freedom are translational (the constant doesn't actually matter that much). We can use this as an estimate for the speed of the exhaust. Thus the gas is expelled at a volumetric rate of $\dot{V} = \pi(r_0+vt)^2\sqrt{\frac{3RT}{M}}$. The density of the gas is $\frac{MP}{RT}$, so the rate of mass expulsion is $\dot{M} = \pi(r_0+vt)^2P\sqrt{\frac{3M}{RT}}$.

This should equal the result from (a) i, yielding $\boxed{P \approx \rho_s\sqrt{\frac{4RT}{3M}}\frac{lv}{r_0+vt}}$

Grading Scheme

- **1 pt** for any dimensionally correct final answer.
- **+2 pts** for a final answer of the correct form (not considering the constant factor).

T2: The Complex Potential

Solution 2:

Part A

Has hinted in the problem, we define differentiability of complex functions just as we would in single variable calculus on the reals. A function $f : \mathbb{C} \to \mathbb{C}$, is complex-differentiable or *holomorphic* at a point $z_0 \in \mathbb{C}$ if the following limit converges to a number $f'(z_0) \in \mathbb{C}$:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \tag{1}$$

We call the number $f'(z_0)$ the derivative of f at z_0. Note that the limit $h \to 0$ is taken by any sequence of points *in the complex plane* that converges to the origin. As in multivariable calculus on $\mathbb{R}^2$, h can approach the origin in two different ways: along the real axis and along the imaginary axis – both limits should yield the same result if the limit in (1) indeed converges. If h moves along the real axis, $h = \delta x \in \mathbb{R}$. The limit in (1) becomes:

$$\lim_{\delta x \to 0} \frac{f(x_0 + \delta x + iy_0) - f(x_0 + iy_0)}{\delta x}$$

where $z_0 = x_0 + iy_0$. If we write $f(x + iy) = w(x, y) + iu(x, y)$, the limit then becomes:

$$\lim_{\delta x \to 0} \left[\frac{w(x_0 + \delta x, y_0) - w(x_0, y_0)}{\delta x} + i \frac{u(x_0 + \delta x, y_0) - u(x_0, y_0)}{\delta x} \right]$$

but we see that these are just partial derivatives evaluated at z_0:

$$\left[\frac{\partial w}{\partial x} + i \frac{\partial u}{\partial x} \right]_{z_0}$$

Now, if we move along the imaginary axis, $h = i\delta y \in i\mathbb{R}$. The limit in (1) becomes:

$$\lim_{\delta y \to 0} \frac{f(x_0 + i(y_0 + \delta y)) - f(x_0 + iy_0)}{i\delta y}$$

Following a similar procedure from before:

$$\lim_{\delta y \to 0} \left[\frac{w(x_0, y_0 + \delta y) - w(x_0, y_0)}{i\delta y} + i \frac{u(x_0, y_0 + \delta y) - u(x_0, y_0)}{i\delta y} \right]$$

Since $\frac{1}{i} = -i$, the limit can be written as:

$$\left[\frac{\partial u}{\partial y} - i \frac{\partial w}{\partial y} \right]_{z_0}$$

As discussed, if f is complex-differentiable at z_0, these limits should be equal. Equating real and imaginary parts, we find:

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned} \tag{2}$$

Part B

From the conditions in (2), we observe that:

$$\begin{aligned} \nabla^2 w &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \\ &= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0 \end{aligned}$$

It can similarly be shown that $\nabla^2 u = 0$. We therefore find that a holomorphic function has real and imaginary parts that satisfy the 2D Laplace equation.

Part C

The 'tangent vectors' along γ_1 at t_1 and γ_2 at t_2 are represented by the complex numbers $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, respectively. Let us denote the 'dot product' of two complex numbers z and w as: $\langle z, w\rangle \equiv \text{Re}[z]\text{Re}[w] + \text{Im}[z]\text{Im}[w]$. The cosine of the angle α is therefore:

$$\cos\alpha = \frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|}$$

The transformed curves $f(\gamma_1(t))$ and $f(\gamma_2(t))$, at the intersection $f(p)$, will have tangent vectors $w_1 = \frac{d}{dt}f(\gamma_1(t))\big|_{t_1}$ and $w_2 = \frac{d}{dt}f(\gamma_2(t))\big|_{t_2}$, respectively. Let us write $\gamma_i(t) = x_i(t) + y_i(t)$ for $i = 1, 2$ and $\tilde{p} = (x_1(t_1), y_1(t_1)) = (x_2(t_2), y_2(t_2)) \in \mathbb{R}^2$. Using the chain rule,

$$\text{Re}[w_1] = \frac{\partial w}{\partial x}\bigg|_{\tilde{p}} \frac{dx_1}{dt}\bigg|_{t_1} + \frac{\partial w}{\partial y}\bigg|_{\tilde{p}} \frac{dy_1}{dt}\bigg|_{t_1}$$
$$\text{Im}[w_1] = \frac{\partial u}{\partial x}\bigg|_{\tilde{p}} \frac{dx_1}{dt}\bigg|_{t_1} + \frac{\partial u}{\partial y}\bigg|_{\tilde{p}} \frac{dy_1}{dt}\bigg|_{t_1}$$

And similarly for w_2:

$$\text{Re}[w_2] = \frac{\partial w}{\partial x}\bigg|_{\tilde{p}} \frac{dx_2}{dt}\bigg|_{t_2} + \frac{\partial w}{\partial y}\bigg|_{\tilde{p}} \frac{dy_2}{dt}\bigg|_{t_2}$$
$$\text{Im}[w_2] = \frac{\partial u}{\partial x}\bigg|_{\tilde{p}} \frac{dx_2}{dt}\bigg|_{t_2} + \frac{\partial u}{\partial y}\bigg|_{\tilde{p}} \frac{dy_2}{dt}\bigg|_{t_2}$$

Now, let us consider the cosine of the angle formed between w_1 and w_2:

$$\frac{\langle w_1, w_2\rangle}{\|w_1\|\|w_2\|}$$

Denoting the t derivatives with dots and omitting the input points/parameter values,

$$\text{Re}[w_1]\text{Re}[w_2] = \left(\frac{\partial w}{\partial x}\right)^2 \dot{x}_1\dot{x}_2 + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}(\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left(\frac{\partial w}{\partial y}\right)^2 \dot{y}_1\dot{y}_2$$
$$\text{Im}[w_1]\text{Im}[w_2] = \left(\frac{\partial u}{\partial x}\right)^2 \dot{x}_1\dot{x}_2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left(\frac{\partial u}{\partial y}\right)^2 \dot{y}_1\dot{y}_2$$

Hence,

$$\langle w_1, w_2\rangle = \left\|\frac{\partial f}{\partial x}\right\|^2 \dot{x}_1\dot{x}_2 + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right](\dot{x}_2\dot{y}_1 + \dot{y}_2\dot{x}_1) + \left\|\frac{\partial f}{\partial y}\right\|^2 \dot{y}_1\dot{y}_2$$

Note that since f is complex-differentiable, we require $\left\|\frac{\partial f}{\partial x}\right\|^2 = \left\|\frac{\partial f}{\partial y}\right\|^2 = \|f'(p)\|^2$. Applying the condition found in (a) to the second term, we find that it vanishes. Hence,

$$\langle w_1, w_2\rangle = \|f'(p)\|^2(\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2) = \|f'(p)\|^2\langle\gamma_1'(t_1), \gamma_2'(t_2)\rangle$$

Now, observe that $w_1 = f'(p)\gamma_1'(t_1)$ and $w_2 = f'(p)\gamma_2'(t_2)$

$$\frac{\langle w_1, w_2\rangle}{\|w_1\|\|w_2\|} = \frac{\langle\gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|}$$

As was to be shown.

Part D

Note that for a holomorphic complex potential $f = \phi + i\psi$, taking the derivative in the real direction:

$$\frac{df}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

ψ was chosen such that $\partial_x\psi = -\partial_y\phi = E_y$ so

$$\frac{df}{dz} = -E_x + iE_y$$

Part E

There was a domain issue in the original problem statement – π should be added/subtracted depending on the domain, but this shouldn't matter in verifying the 2D Laplace equation since it involves derivatives. A better statement would be $\phi(r,\theta) = \frac{\phi_0}{\pi}\theta$ for $0 \le \theta \le \pi$, which clearly satisfies the given boundary conditions $\phi(x > 0, 0) = 0$, $\phi(x < 0, 0) = \phi_0$. In spherical coordinates, the Laplacian operator is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Hence,

$$\nabla^2 \phi(r,\theta) = 0$$

By the uniqueness theorem, the true, unique electric field respecting the boundary conditions is modeled by $-\nabla\phi$.

Part F

A radial line on the upper half plane at angle $\theta \in (0,\pi)$ is the set of points $L_\theta = \{re^{i\theta} \in \mathcal{H}, r \in (0,\infty)\}$. Under the logarithm map, this set is mapped to $\log(L_\theta) = \{\log|r| + i\theta, r \in (0,\infty)\}$ which is clearly the horizontal line at $\mathrm{Im}[z] = \theta$ extending $-\infty < \mathrm{Re}[z] < \infty$. Since the upper half plane can be thought as the collection of the sets L_θ for $\theta \in (0,\pi)$, $\log\mathcal{H}$ is clearly the strip bounded by $\mathrm{Im}[z] = 0$ and $\mathrm{Im}[z] = \pi$. Let us call this strip $\mathcal{S}$. We are given that $\log(z)$ is holomorphic on $\mathcal{H}$ and that $e^z : \mathcal{S} \to \mathcal{H}$

Part G

Note that the boundary condition in (e), under the logarithm map, becomes the desired boundary conditions of two infinite capacitor plates. Let us crudely denote the 'potential space' as Φ – the space to which complex potentials map. For example, if f is an appropriate complex potential to (e), we may write $f : \mathcal{H} \to \Phi$ with the boundary condition $\phi(x > 0, 0) = 0$ and $\phi(x < 0, 0) = \phi_0$ defined on the upper half plane $\mathcal{H}$. We observe that taking a map $f \circ e^z : \mathcal{S} \to \mathcal{H} \to \Phi$ defines a complex potential with the boundary conditions $\phi(x, 0) = 0$ and $\phi(x,\pi) = \phi_0$ defined on $\mathcal{S}$. For an arbitrary separation d, we can consider $f(e^{\frac{\pi}{d}z})$. Since $e^{\frac{\pi}{d}z}$ and f are holomorphic, so is $f(e^{\frac{\pi}{d}z}) = \phi(e^{\frac{\pi}{d}z}) + i\psi(e^{\frac{\pi}{d}z})$. As verified in (b), this means that the real part, $\phi(e^{\frac{\pi}{d}z})$, satisfies the 2D Laplace equation, as well as the boundary conditions $\phi(x, 0) = 0$ and $\phi(x, d) = \phi_0$, hence a valid potential for this problem. We have:

$$\phi_c(x,y) = \phi\left(e^{\frac{\pi}{d}r\cos\theta}, \frac{\pi}{d}r\sin\theta\right) = \frac{\phi_0}{d}r\sin\theta = \frac{\phi_0}{d}y$$

for $y \in (0, d)$. This is indeed the expected result from basic electrostatics.

Part H

This problem really isn't approachable without rigorous mathematical reasoning. The unit disc is the set defined by $\{z \in \mathbb{C}, |z| < 1\}$. The upper half plane is defined by the set $\{z \in \mathbb{C}, \mathrm{Im}[z] \ge 0\}$. We first show that $f(\mathcal{D}) \subseteq \mathcal{H}$. For any $\xi \in \mathcal{D}$, we observe that

$$\begin{aligned}
\mathrm{Im}\left[i\frac{1-\xi}{1+\xi}\right] &= \mathrm{Re}\left[\frac{1-\xi}{1+\xi}\right] \\
&= \mathrm{Re}\left[\frac{(1-\xi)(1+\xi^*)}{\|1+\xi\|^2}\right] \\
&= \mathrm{Re}\left[\frac{1+\xi^*-\xi-\|\xi\|^2}{\|1+\xi\|^2}\right] \\
&= \mathrm{Re}\left[\frac{1-2i\mathrm{Im}[\xi]-\|\xi\|^2}{\|1+\xi\|^2}\right] \\
&= \frac{1-\|\xi\|^2}{\|1+\xi\|^2}
\end{aligned}$$

Clearly, if $|\xi| < 1$, we have

$$\operatorname{Im}\left[i\frac{1-\xi}{1+\xi}\right] > 0$$

implying $f(\xi) \in \mathcal{H}$. Since for an arbitrary $f(\xi) \in f(\mathcal{D})$ we have $f(\xi) \in \mathcal{H}$, $f(\mathcal{D}) \subseteq \mathcal{H}$. Now we'd like to show that $\mathcal{H} \subseteq f(\mathcal{D})$. Consider an arbitrary $\omega \in \mathcal{H}$. We'd like to show that there exists a number $\xi \in \mathcal{D}$ such that $f(\xi) = \omega$, which would imply $\omega \in f(\mathcal{D})$. Observe that:

$$\begin{aligned} i\frac{1-\xi}{1+\xi} &= \omega \\ i - i\xi &= \omega + \omega\xi \\ i - \omega &= \xi(\omega + i) \\ \xi &= \frac{i-\omega}{i+\omega} \end{aligned}$$

Now, note that $\operatorname{Re}[i-\omega] = -\operatorname{Re}[i+\omega] = \operatorname{Re}[\omega]$, so when comparing $|i-\omega|$ and $|i+\omega|$, we need only compare the imaginary parts. Denoting $\omega_i \equiv \operatorname{Im}[\omega] > 0$,

$$|\operatorname{Im}[i+\omega]| = |1+\omega_i| \geq |1-|\omega_i||$$

since $\omega_i > 0$, we can get rid of the absolute value symbol – $|\omega_i| = \omega_i$ – and make the above a strict inequality. This suggests that $|\operatorname{Im}[i-\omega]| < |\operatorname{Im}[i+\omega]|$, which implies $|i-\omega| < |i+\omega|$, and we conclude:

$$|\xi| = \left|\frac{i-\omega}{i+\omega}\right| < 1$$

so for all $\omega \in \mathcal{H}$, there exists a $\xi \in \mathcal{D}$ with $f(\xi) = \omega$. We've shown that $f(\mathcal{D}) \subseteq \mathcal{H}$ and that $f(\mathcal{H}) \subseteq \mathcal{D}$, implying $f(\mathcal{D}) = \mathcal{H}$. The way we found ξ above hints that the $\omega \in \mathcal{H}$, $\xi \in \mathcal{D}$ pair is defined uniquely – that there is a one-to-one correspondence such that f is bijective. To show this, we can first suppose that f has an inverse h defined by:

$$h(w) = \frac{i-w}{i+w}$$

h is clearly single-valued, and it's easy to verify that $h(f(z)) = z$ and $f(h(w)) = w$. This would be a direct proof that f is a bijection between $\mathcal{D}$ and $\mathcal{H}$. Now we observe the mapping of the circle $C = \{|z| = 1\}$. Note that there should be problems with f around $z = -1 \in C$ since f is not holomorphic at $z = -1$. For an arbitrary point $e^{i\theta} \in C$ with $\theta \in [0, 2\pi)$, it's easy to simplify expressions down to:

$$f(e^{i\theta}) = \frac{\sin\theta}{1+\cos\theta}$$

Graphing this on desmos suggests that the arc $(0,\pi)$ maps to the positive real line and that $(\pi, 2\pi)$ maps to the negative real line. Notice there's a jump from ∞ to $-\infty$ as we move from the top arc to the bottom arc at π – this is where f "breaks".

Part I

This is a standard electrostatics problem. The potential ϕ_I in the region between the rod and the cylindrical shell can be found by integrating the appropriate electric field $\boldsymbol{E}$.

$$\begin{aligned} \boldsymbol{E}(r) &= \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \\ \phi_I(r) &= -\frac{\lambda}{2\pi\epsilon_0}\ln(r/R) \end{aligned}$$

where R is the radius of the concentric cylindrical shell.

Part J

From our results in (h), let us define a map $g : \mathcal{D}_R \to \mathcal{H}$, where $\mathcal{D}_R$ is the disc of radius R centered at the origin, as:

$$g(z) = ih\frac{1 - z/R}{1 + z/R}$$

Under this map, the set up in (i) is mapped to the boundary conditions on $\mathcal{H}$: $\phi = 0$ on the real line, infinite rod placed at $z = ih$. These are the exact boundary conditions that we require for this problem. The inverse map of g, denoted H, is given by:

$$H(w) = R\frac{i - w/h}{i + w/h}$$

H is holomorphic everywhere on the complex plane except at $-ih$, which isn't included in $\mathcal{H}$, so H holomorphic on the upper half plane. Since the boundary condition satisfied by ϕ_I has the geometry of $\mathcal{D}_R$, we can consider a complex potential $f = \phi_I + i\psi_I$ that maps $f : \mathcal{D}_R \to \Phi$. As argued in (g), $f \circ H : \mathcal{H} \to \mathcal{D}_R \to \Phi$ is a holomorphic function on $\mathcal{H}$ that respects all necessary boundary conditions. Since ϕ_I is only dependent on r, we'd like to consider $|H(w)|$:

$$\|H(\omega)\|^2 = R^2\frac{1 - \frac{2}{h}\,\mathrm{Im}[w] + \frac{\|w\|^2}{h^2}}{1 + \frac{2}{h}\,\mathrm{Im}[w] + \frac{\|w\|^2}{h^2}}$$

If we identify $\mathrm{Im}[w] = y$ and $\|w\|^2 = x^2 + y^2$, the desired potential function is:

$$\phi(x, y) = \frac{\lambda}{2\pi\epsilon_0}\ln\sqrt{\frac{1 + \frac{2}{h}y + \frac{x^2+y^2}{h^2}}{1 - \frac{2}{h}y + \frac{x^2+y^2}{h^2}}}$$

A method of images results in the same expression.

Part K

If we parameterize the curve C as $\gamma(t) = x(t) + iy(t)$, the length element dl at $\gamma(t_0)$ can be written as:

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = |\gamma'(t_0)|dt$$

with all derivatives evaluated at t_0. We can use the calculations in (c) to write dl' in terms of dt:

$$dl' = \left|\frac{d}{dt}f(\gamma(t))\Big|_{t_0}\right|dt = |f'(\gamma(t_0))||\gamma'(t_0)|dt$$

Hence, we find the scaling factor is $|f'(\gamma(t_0))|$.

Part L

If $\hat{n}$ is orthogonal to dl, the surface charge at dl is proportional to $\nabla\phi \cdot \hat{n}$. Since the surface is a conductor, $\nabla\phi$ is parallel to $\hat{n}$ near the surface. From previous problems, we know that $\phi' = \phi \circ f^{-1}$ is the appropriate electrostatic potential in $f(C)$, and $\nabla\phi'$ is parallel to the mapping of $\hat{n}$, which remains orthogonal to the curve, as shown in (c). There's a theorem that states that if f is a bijective holomorphic map, f' never vanishes, f^{-1} is holomorphic, and the derivative of f^{-1} is given by $\frac{1}{f'}$. However, since we deal with the individual real and imaginary components of complex functions in this question, we may crudely rely on calculus on $\mathbb{R}^2$. The surface charge at dl' is proportional to $\nabla\phi' \cdot \hat{n}' = |\nabla\phi'|$. Carefully applying the chain rule results in:

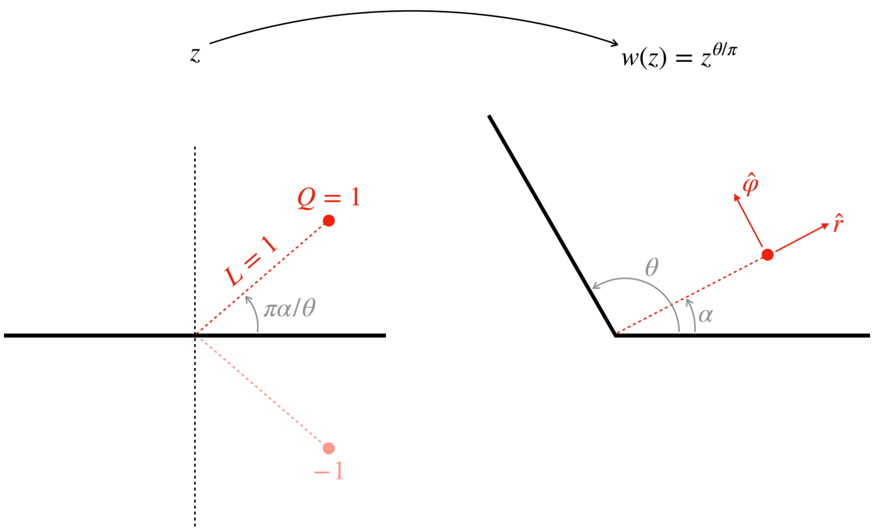
$$|\nabla\phi'| = \frac{|\nabla\phi|}{|f'(z)|}$$

Part M

If we take the charge at points to remain the same even after conformal transformations (think about stretching charged surfaces with the charges glued in place) we see that the scale factor found in (j) results in the scaling of the surface charge found in (l). Hence, given that functions behave nicely at boundaries, we can determine the surface charge of other surfaces simply by looking at the form of f and its derivative.

Part N

The electrostatic system of interested is effectively two-dimensional. For simplicity, set $L = 1$ and $Q = +1$. We can recover the general answer with dimensional analysis later on.
Consider a configuration in z-space in which the grounded conductor fills the area $\text{Re}(z) < 0$ and a charge is placed at position $e^{+i\pi\alpha/\theta}$. This configuration can be mapped to our puzzle in w-space by the conformal transformation $w(z) = z^{\theta/\pi}$ as shown in the figure below.



In z-space, the electric potential in open space is as if created by the original charge and the image charge -1 located at position $e^{-i\pi\alpha/\theta}$. Thus, the complex potential is given by:

$$\phi(z) = \left[-2k\ln\left(z - e^{+i\pi\alpha/\theta}\right)\right] - \left[-2k\ln\left(z - e^{-i\pi\alpha/\theta}\right)\right] = 2k\ln\frac{z - e^{-i\pi\alpha/\theta}}{z - e^{+i\pi\alpha/\theta}} .$$

After the transformation $z \to w(z)$, in w-space it becomes $\phi(z) \to \phi'(w)$:

$$\phi'(w) = \phi(z)\Big|_{z\to w} = 2k\ln\frac{w - e^{-i\pi\alpha/\theta}}{w - e^{+i\pi\alpha/\theta}} .$$

Choose a polar coordinate $w = (r, \varphi)$, then we can obtain the real potential as:

$$\text{Re}\left[\phi'(r,\varphi)\right] = k\ln\frac{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi+\alpha)\right]}{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi-\alpha)\right]} .$$

Subtract this by the electric potential created by the original charge in the w-space, we have a regularized complex potential which is not singular at the location of that charge:

$$\begin{aligned}\text{Re}\left[\phi'_{\text{reg}}(r,\varphi)\right] &= \text{Re}\left\{\phi'(r,\varphi) - \left[-2k\ln\left(w - e^{i\alpha}\right)\right]\Big|_{w=re^{i\varphi}}\right\} \\ &= k\ln\frac{\left\{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi+\alpha)\right]\right\}\left[1 + r^2 - 2r\cos(\varphi-\alpha)\right]}{1 + r^{2\pi/\theta} - 2r^{\pi/\theta}\cos\left[\frac{\pi}{\theta}(\varphi-\alpha)\right]} .\end{aligned}$$

We can then calculate the electrical field created at the location of the original charge by the induced charge distributed on the conducting wedge:

$$\vec{E} = -\left\{\partial_r \mathrm{Re}\left[\phi'_{\mathrm{reg}}(r,\varphi)\right]\hat{r} + \frac{1}{r}\partial_\varphi \mathrm{Re}\left[\phi'_{\mathrm{reg}}(r,\varphi)\right]\hat{\varphi}\right\}\Bigg|_{r=1,\varphi=\alpha} = -k\left[\hat{r} + \frac{\pi}{\theta}\cot\left(\frac{\pi}{\theta}\alpha\right)\hat{\varphi}\right] \ .$$

The total electrostatic force acting on the original charge is therefore $\vec{F} = \vec{E}$, and after putting back Q and L, we arrive at the expression:

$$\vec{F} = -k\frac{Q^2}{L}\left[\hat{r} + \frac{\pi}{\theta}\cot\left(\frac{\pi}{\theta}\alpha\right)\hat{\varphi}\right] \ .$$

For $\theta = 120^o$ and $\alpha = 30^o$, the magnitude of this force in the unit of kQ^2/L is $\sqrt{13}/2 \approx 1.8028$.

* This part was created with helps from Quy C. Tran and Nam H. Nguyen.

Grading Scheme

(a) 1.5 pts
(b) 1.5 pts
(c) 5 pts
(d) 1.5 pts
(e) 1 pt
(f) 1.5 pts
(g) 7.5 pts
(h) 7.5 pts
(i) 1.5 pts
(j) 7.5 pts
(k) 5 pts
(l) 5 pts
(m) 1.5 pts
(n) 15 pts

T3: General Relativity

Solution 3:

(a) By the multivariable chain rule, we can write the relationship between x_i and q_i as

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial q_1}\mathrm{d}q_1 + \frac{\partial x_i}{\partial q_2}\mathrm{d}q_2 + \cdots + \frac{\partial x_n}{\partial q_n}\mathrm{d}x_n = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j.$$

We know that in original coordinates, the infinitesimal line element can be written by the Pythagorean theorem as

$$\mathrm{d}s = \sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2} \implies \mathrm{d}s^2 = \sum_{i=1}^{n} \mathrm{d}x_i^2$$

Hence, we can find that

$$\mathrm{d}s^2 = \sum_{i}^{n} \left(\frac{\partial \vec{x}}{\partial q_i}\right)^2 \mathrm{d}q_i.$$

This means that $g_i = \left(\frac{\partial \psi^{-1}}{\partial q_i}\right)^2$.

Grading Scheme

- **(1 pt)** Uses multivariable chain rule or equivalent to find infintesimal length $\mathrm{d}x_i$.
- **(1 pt)** Uses Pythagorean theorem to find formula for line element $\mathrm{d}s$.
- **(1 pt)** Finds out that $g_i = \left(\frac{\partial \psi^{-1}}{\partial q_i}\right)^2$

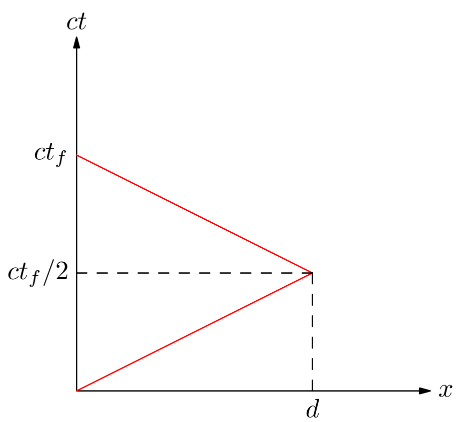
Notes

- We do not expect competitors to provide the most mathematically rigorous solution. A solution with true rigor would be able to prove why $\mathrm{d}s^2$ takes the given form.
- Some competitors were able to provide a solution explaining this. See the solution by $|\text{Enloe}\rangle$ on our website.

(b) Let us take a look at the Minkowski metric:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Without loss of generalization, let us assume the rocket is moving only in the x-direction, meaning that $\mathrm{d}y = \mathrm{d}z = 0$. We can draw a spacetime diagram as shown below for the twin moving in the rocket:



Note that $\mathrm{d}x = v\mathrm{d}t$ where v is the constant velocity of the twin. Hence, we can rewrite as

$$\mathrm{d}s^2 = (v^2 - c^2)\mathrm{d}t^2.$$

Since $\mathrm{d}s^2$ is invariant, we can say that the time experienced by the twin on the spaceship is τ. As the other twin is not moving, then (and by noting that v changes sign at $ct_f/2$)

$$-c^2\mathrm{d}\tau^2 = (v^2 - c^2)\mathrm{d}t^2 \implies \Delta\tau = \int_0^{t_f} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t \implies \tau = \frac{t_f}{\gamma}.$$

Hence the twin on the spaceship ages less.

Grading Scheme

- **(0.5 pts)** Writes the relation $\mathrm{d}x = v\mathrm{d}t$.
- **(1 pt)** Uses invariance to find τ in terms of t. (-0.5 points if they do not acknowledge the change in sign of v at $ct_f/2$)
- **(0.5 pts)** Concludes that the spaceship twin ages less.

(c) We recall that the action of a system along a path $\vec{q}(t)$ between two times t_1 and t_2 is given as

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t.$$

We are given that the line element $\mathrm{d}s^2 = -(1 + 2\Phi(x))\mathrm{d}t^2 + (1 - 2\Phi(x))\mathrm{d}x^2$ (using units where $c = 1$). We can write an expression for the maximal spacetime length as

$$\begin{aligned}
\mathrm{d}s &= \sqrt{-(1 + 2\Phi(x))\mathrm{d}t^2 + (1 - 2\Phi(x))\mathrm{d}x^2} \\
&= \sqrt{-(1 + 2\Phi(x))\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)^2 + (1 - 2\Phi(x))\left(\frac{\mathrm{d}x}{\mathrm{d}\sigma}\right)^2}\mathrm{d}\sigma \\
S = \int \mathrm{d}s &= \int \sqrt{-(1 + 2\Phi(x)) + (1 - 2\Phi(x))\dot{x}^2}\mathrm{d}\sigma
\end{aligned}$$

We notice that the expression inside the integral is the Lagrangian. Now we use the Euler-Lagrange equation $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{x}}$ using the approximation $\Phi(x) \ll 1$ and $\dot{x} \ll 1$. Without much detail on calculations, one can recover that $\ddot{x} = -\Phi'(x)$. This resembles Newton's second law $F = -m\frac{\mathrm{d}\Phi}{\mathrm{d}x}$.

Grading Scheme

- **(2 pts)** Is able to recover the Lagrangian. (-0.5 points if they write the integral with $\mathrm{d}t$ instead of $\mathrm{d}\sigma$.)
- **(2 pts)** Is able to recover Newton's second law by using the Euler-Lagrange equations. (-1 point if significant progress is achieved, but the final answer is not correct. -0.5 points if they don't show how they used approximations)

(d) To get rid of special relativistic effects, we assume zero velocity, and hence $\mathrm{d}s^2 = -(1+2\Phi(x))\mathrm{d}t^2$. As the factor for time dilation is $\gamma = \sigma/t$, we can write using $\mathrm{d}s^2 = -\mathrm{d}\sigma^2$ that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sqrt{1 + 2\Phi(x)} \approx 1 + \Phi(x)$$

Grading Scheme

- **(1 pt)** Writes expression assuming $\mathrm{d}s^2$ for general relativistic effects.
- **(1 pt)** Finds expression for time dilation.

(e) We apply the radial geodesic. We note that $\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0$. Thus, we can add the geodesic equations for Γ^r_{tt} and $\Gamma^r_{\phi\phi}$, so that

$$\Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0$$

$$\left(\frac{d\phi}{dt}\right)^2 = -\frac{\Gamma^r_{tt}}{\Gamma^r_{\phi\phi}} = -\frac{\frac{c^2 B \frac{\mathrm{d}B}{\mathrm{d}r}}{2}}{-Br\sin^2\theta} = \frac{c^2 \frac{\mathrm{d}B}{\mathrm{d}r}}{2r} = \frac{c^2 r_s}{2r^3} = \frac{GM}{r^3}$$

Thus,

$$\left(r\frac{d\phi}{dt}\right)^2 = c^2 - \frac{2GM}{r} = \frac{GM}{r},$$

which yields $r_p = \frac{3GM}{c^2}$.

Grading Scheme

- **(2 pts)** Recombines the radial geodesic equation to find a formula for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$.
- **(2 pts)** Finds the radius to be $r_p = \frac{3}{2}r_s$. (-1 pt if significant progress is achieved but the final answer is not correct)

(f) Divide the Schwarzschild metric by the path parameter to obtain:

$$0 = -\left(1 - \frac{r_s}{r}\right)c^2 t^2 + \frac{\dot{r}^2}{1 - \frac{r_s}{r}} + r^2\dot{\phi}^2$$

Where $ds^2 = 0$ for null geodesics. From this, we can identify the following conserved quantities:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(r^2\dot{\phi}\right) \Rightarrow L = r^2\dot{\phi} \quad \text{(angular momentum)}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left(1 - \frac{r_s}{r}\right)c^2\dot{t}\right) \Rightarrow E = \left(1 - \frac{r_s}{r}\right)c^2\dot{t} \quad \text{(energy)}$$

Here, $b = \frac{L}{E}$ is the impact parameter. The Schwarzschild metric, for $\theta = \frac{\pi}{2}$, can be rewritten as:

$$0 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\phi^2$$

This leads to:

$$E^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{r_s}{r}\right)$$
$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = -\left(1 - \frac{r_s}{r}\right) + \frac{r^4}{b^2}$$

Differentiating with respect to ϕ and substituting $1/r$ shows:

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) = \frac{r_p}{r^2} - \frac{1}{r}$$

Considering a small perturbation $r = r_p \pm \delta$ leads to:

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r+\delta}\right) = \frac{r_p}{(r+\delta)^2} - \frac{1}{r+\delta}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\left(1 + \frac{\delta}{r}\right)^{-1}\right) = \frac{r_p}{r^2}\left(1 + \frac{\delta}{r}\right)^{-2} - \frac{1}{r}\left(1 + \frac{\delta}{r}\right)$$
$$-\frac{1}{r^2}\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2} = \frac{1}{r}\left[\left(1 - 2\frac{\delta}{r}\right) - \left(1 + \frac{\delta}{r}\right)\right]$$

Finally, we can retrieve a simple differential equation:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2} = \delta \Rightarrow \delta = Ae^{\phi} + Be^{-\phi}$$

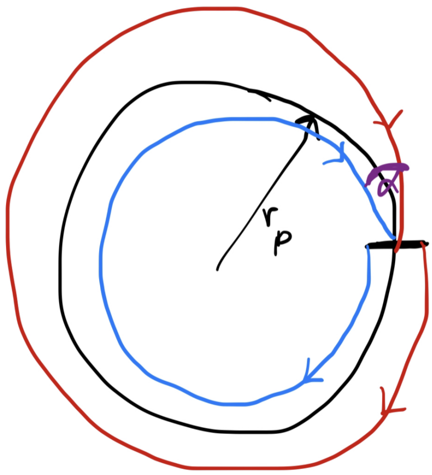
Given $\delta(0) = \delta_0$ (e.g., $\delta = 0.25$ m) and $\delta'(0) = r_p\beta$ (where β is an arbitrary constant representing the angle), one can find A and B as

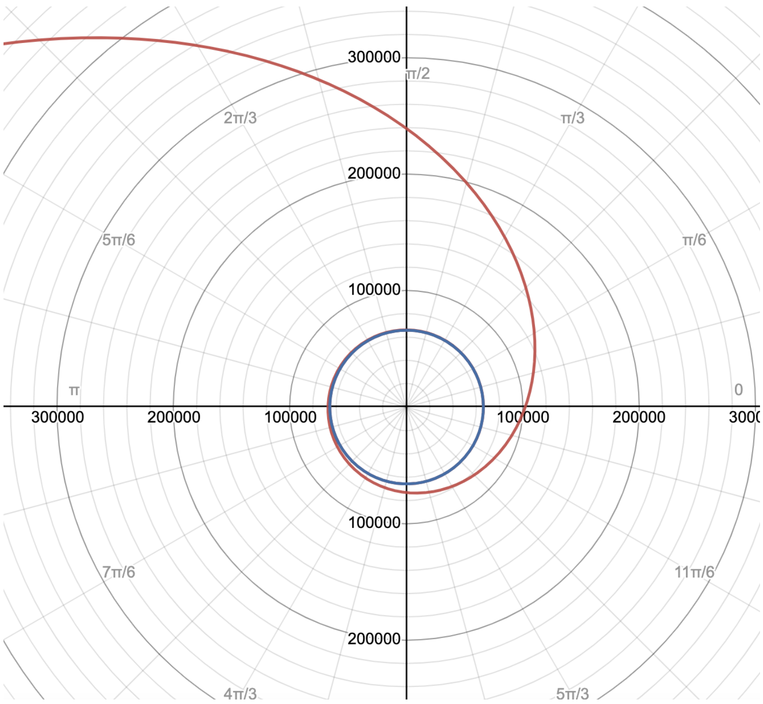
$$\begin{aligned}\delta &= \delta_0 \left(\frac{e^{\phi} + e^{-\phi}}{2}\right) + r_p^{\alpha}\left(\frac{e^{\phi} - e^{-\phi}}{2}\right)\\ &= \delta_0 \cosh\phi + r_p\beta\sinh\phi\end{aligned}$$

Thus, for the maximum case, we need $\delta(2\pi) = \delta_0$. To find α, consider the symmetry and multiply the result by 2. Therefore,

$$\delta_0 = \delta_0\cosh(2\pi) + \frac{r_p\alpha}{2}\sinh(2\pi) \implies \alpha = \frac{2\delta_0(1-\cosh(2\pi))}{r_p\sinh(2\pi)}$$

A drawing can be created on desmos.





Notice how the angle k must be extremely small for the effect to occur. In the above picture we set $k = 0.00000002$ and yet, the photon trajectory misses by about 3000 meters when we require the full deviation to be 0.25 m. Hence, it is extremely unlikely for one to be able to visualize this effect in real life. Additionally, it is worth noticing how these curves take form of spirals.

Grading Scheme

- **(2 pts)** For recovering expressions for angular momentum and energy.

- **(2 pts)** For finding a differential equation in terms of r_p, r, and ϕ.
- **(4 pts)** For simplifying the differential equation and using proper approximation to find an expression for $\frac{\mathrm{d}^2\delta}{\mathrm{d}\phi^2}$.
- **(2 pts)** For solving the differential equation and interpreting the results.
- **(1 pts)** For creating a proper labelled diagram that represents the scenario.

(g) The Schwarzchild Metric is represented by:

$$(ds)^2 = \left(cdt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - \left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2 - (r\ d\theta)^2 - (r\sin\theta d\phi)^2$$

as we are only worried about the ϕ direction, $dr = d\theta = 0$ and $\theta = \pi/2$. Therefore, we have

$$c\sqrt{1-\frac{2GM}{rc^2}} = r\ \frac{d\phi}{dt}$$

To find the period, we need to calculate the angular speed of light in the ϕ-direction at $\frac{3GM}{c^2}$. Therefore,

$$\frac{d\phi}{dt} = \frac{c}{r}\sqrt{1-\frac{2GM}{rc^2}} = \frac{c^3}{3GM\sqrt{3}}.$$

Therefore, because angular speed is constant,

$$T = \frac{2\pi}{\frac{d\phi}{dt}} = \frac{2\pi}{\frac{c^3}{3GM\sqrt{3}}} = \frac{6\pi GM\sqrt{3}}{c^3}$$

Grading Scheme

- **(2 pts)** For simplifying the Schwarzschild metric and finding a formula for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$.
- **(2 pts)** Getting the right answer.

(h) **Grading Scheme**

- **(4 pts)** Solid explanation proving point.

(i) **Grading Scheme**

- **(3 pts)** Finds the radius of the shadow disc is $\frac{3\sqrt{3}}{2}r_s$.
- If part (f) was not attempted, +2 points for recovering expressions for angular momentum and energy.
- **(3 pts)** +1.5 Uses solid angle Ω to find portion of light shining on disc. + 1.5 Integrates using number density to find total number of contributing photons. (-1 pt if correct answer is not obtained.)