2021 Online Physics Olympiad: Invitational Contest

Experimental Exam

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General Instructions

The experimental examination consists of 1 long answer question worth 50 points over 1 full day from August 15, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting.

Specific Rules

For any part of this paper, you are allowed to use online tools and resources to help you, as long as you are not requesting help from anyone outside of your team. Allowable resources include Wikipedia, research papers, Wolfram Alpha, Python, Excel, etc.

However, you must document every resource that you use and cite them when applicable. As a general rule of thumb, you should derive any results that cannot be found on Wikipedia. Therefore, solutions along the lines of: "By Wolfram Alpha, this is true." will not be accepted. Be reasonable please.

Every time you are asked to run an experiment, you must provide the input parameters and a screenshot of the output.

Accessing the Program

To access the Python notebook, follow this **link**. You will be able to perform all the code online, without downloading anything. If you cannot access the link, we will also provide the source code on our website.

Background Information

In this problem, you will use a computer simulation written in *Python* to complete a series of questions relating to slit interference patterns. While you do not need to understand how exactly the code produces the results, it may be beneficial to understand the algorithm the code takes.

The program uses a simple algorithm to determine the interference pattern.

1. An aperture pattern is created by specifying `slits`.
2. The slits are divided into uniform $\lambda \times \lambda$ segments.
3. For each position x on the screen, a wave is generated (and represented as a phasor) between the point x and the center of *each* segment created in the previous step.
4. These phasors are added up and squared to get the intensity.
5. The final intensity pattern is normalized such that the maximum value is 1.

For this entire problem, assume that the heights of the slits are λ.

Remark: This algorithm treats the $d = \lambda$ case as a thin slit (i.e. point source), and it is inaccurate for wide slits (try to see why). This only causes an issue for problem 1.2 & 1.3. Thin slits are used for all other experimental parts afterwards.

Part One

Let us first gain confidence in using the program. To do so, we will derive

$$\frac{\sin(x)}{x} = \cos(x/2)\cos(x/4)\cos(x/8)\cdots \tag{1}$$

via a series of questions.

Problem 1.1 (1 pt)

Suppose monochromatic, coherent light of wavelength λ falls down onto two slits of width w, with midpoints separated by a distance d. Find the amplitude function $A(\theta)$ for the interference pattern produced as a result, where θ is the deviation angle from the center. Assume the screen is far away. A proof is not necessarily necessary, but will help with partial credit in the event that your answer is wrong.

Problem 1.2 (4 pts)

Currently, the code simulates the interference patterns of two thin slits separated by a distance of $d = 8\lambda$. Modify the code to simulate:

- A double slit experiment with wide slits. You are free to choose the location and widths of the slits. Take into account the remark in the background information.

The program will output the intensity function. Make note of the minimum and maximum in the intensity in the experimental results, and compare it to the theoretical results (which you derived in the previous question). Do they agree?

If they do not agree, provide possible reasoning to why they do not agree.

Problem 1.3 (3 pts)

Suppose we have some pattern of slits with overall width w that produces an interference pattern with amplitude $A_0(\theta)$. Suppose we place two of these patterns with midpoints separated a distance $d \geq w$ apart (so that they do not overlap). Find, with proof, as a function of A_0 and other parameters, the amplitude function $A(\theta)$ for the interference pattern produced as a result.

Verify this result experimentally using the code.

Problem 1.4 (5 pts)

Complete the problem by showing

$$\frac{\sin(x)}{x} = \cos(x/2)\cos(x/4)\cos(x/8)\cdots \tag{2}$$

There is no experimental portion associated with this part.

Part Two

Problem 2.1 (8 pts)

Recall that the intensity of the interference pattern from two thin slits behaves like $\cos^2(x)$. Is it possible to have a series of thin slits such that the *amplitude* pattern behaves exactly like the intensity pattern from the two thin slits case? Specifically, the amplitude $A_1(x)$ from one pattern of slits behaves like the intensity $I_2(x)$ from another pattern of slits if:

$$\frac{I_2(x)}{I_{2,\max}} = \frac{A_1(x)}{A_{1,\max}} \tag{3}$$

What sort of aperture would create such an $A(x)$? Verify this experimentally. You will notice that the pattern will deviate towards the edge. At what angle does this deviation become significant? Note that "significant" is subjective, so you will need to provide justification for how you define significant.

Solve this problem using Fourier Optics (There are at least 2 separate ways to do so. As long as you borrow concepts from Fourier Optics, you will receive full points):

- Here are four potentially useful references from Wikipedia. **Any result you use that is not in these references must be derived. This holds for the following two problems as well.**
  - Fraunhofer Diffraction Equation
  - Fourier Optics
  - Convolution
  - Fourier Transform

  Remember that asking for help on public forums or seeking help from other students, other teams, professors, i.e. is strictly prohibited.

Problem 2.2 (4 pts)

In the previous question, you have constructed a series of thin slits such that the amplitude behaves like $\cos^2(x)$. As a result, the intensity behaves like $\cos^4(x)$.

Now, construct a series of slits such that the amplitude function behaves like the intensity pattern from the previous question, i.e. $\cos^4(x)$. You may choose to verify this experimentally, but it is not necessary to get full marks.

Problem 2.3 (4 pts)

Now generalize this to an arbitrary $\cos^n(x)$. If you want the amplitude function to behave like $\cos^n(x)$ where n is a non-negative integer, what should the slit pattern be?

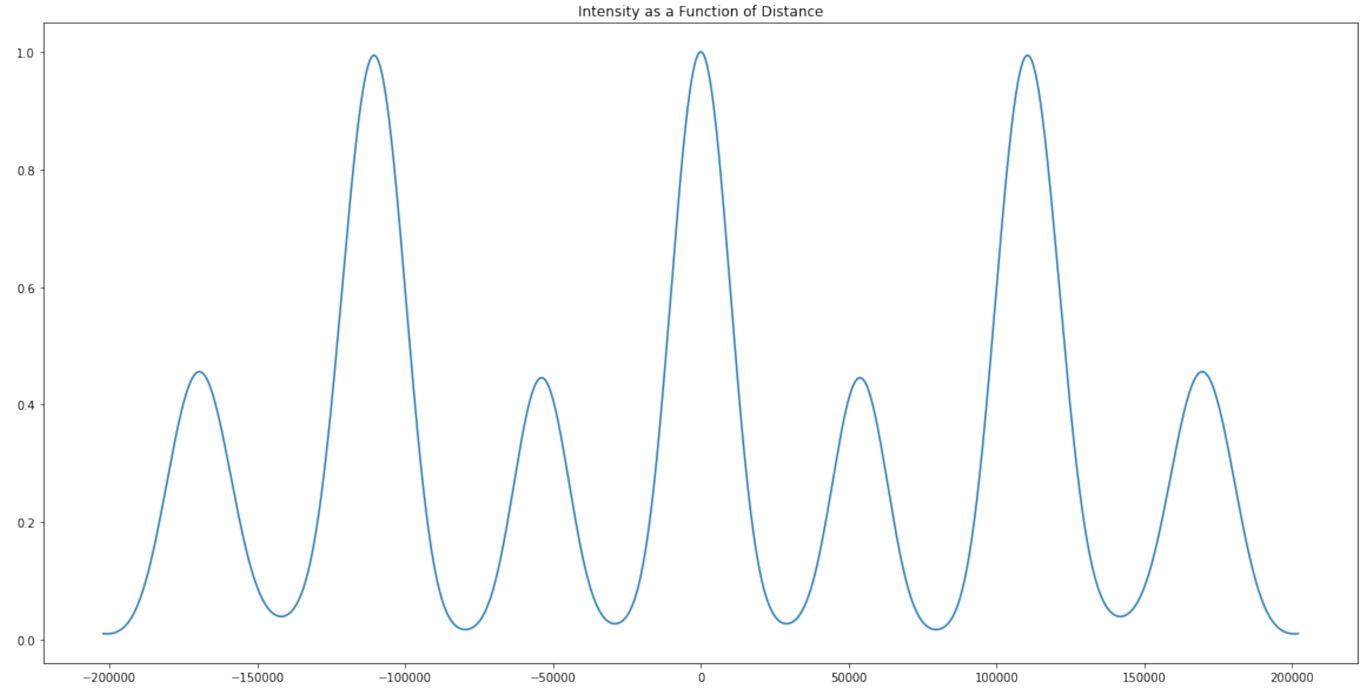
Hint: If you are stuck, try coming up with a conjecture based off of the $\cos^2(x)$ and $\cos^4(x)$ cases, list out how much amplitude they let in, and have everything share a common denominator. If you did the previous parts correctly, the numerators should follow a very familiar pattern.

Part Three

In the previous two parts, you have mostly been asked questions that can be solved analytically, and then used the code to double check your answer. Now, we will ask a few questions that requires data analysis.

Problem 3.1 (11 pts)

The following diagram shows the intensity of the interference pattern produced by a series of slits (where each slit can reduce the amplitude by some factor). The wall is 25 cm away and 500 nm light is used. The x-axis represents locations on the screen, with units of λ.

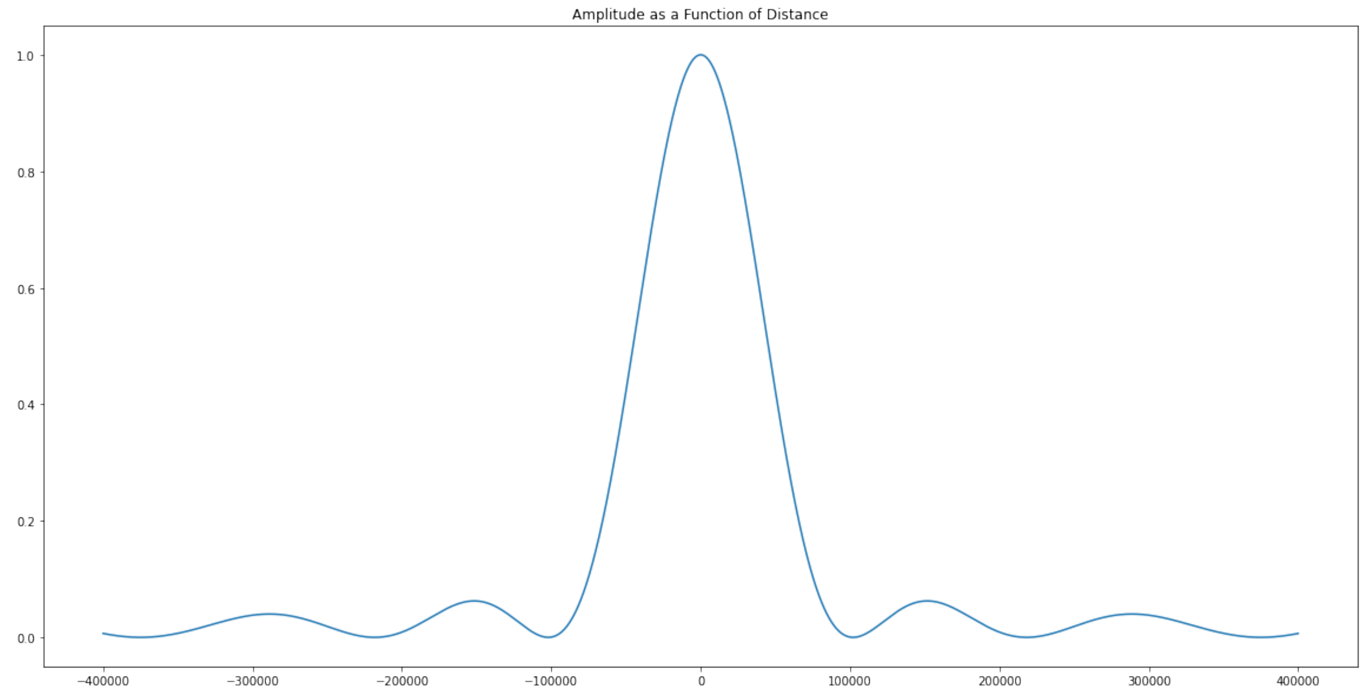


Determine the aperture pattern to the best of your ability. The raw data file is attached on the website.

Note: You may use any online tool and/or resource to do this problem, so long as you are not asking help from people outside your team.

Problem 3.2 (7 pts)

The following diagram shows the **amplitude** of the interference pattern produced by a series of 9 thin slits, centered at $-4\lambda, -3\lambda, \ldots, +4\lambda$. Each slit reduces the wavelength by some factor (this factor isn't necessarily the same for each slit).



Again, the screen is the same distance away as the previous problem and the same scale is used. Determine how much each slit reduces the amplitude of light that passes through.

Let f_x be the reduction factor of the slit centered at x, which has units of λ. For example, the rightmost slit is located at $x = 4$. Make a plot of (x, f_x) for $0 \leq x \leq 4$ and make note of patterns you see.

Problem 3.3 (3 pts)

The interference pattern from the previous problem can be approximated (at least in the small angle range) by a relatively simple function. Find this function.

Hint: Look at the pattern that was hinted at in the previous question

Solution

Part One

Problem 1.1

We have:

$$A(\theta) \propto \left(\frac{\sin\beta}{\beta}\right)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{4}$$

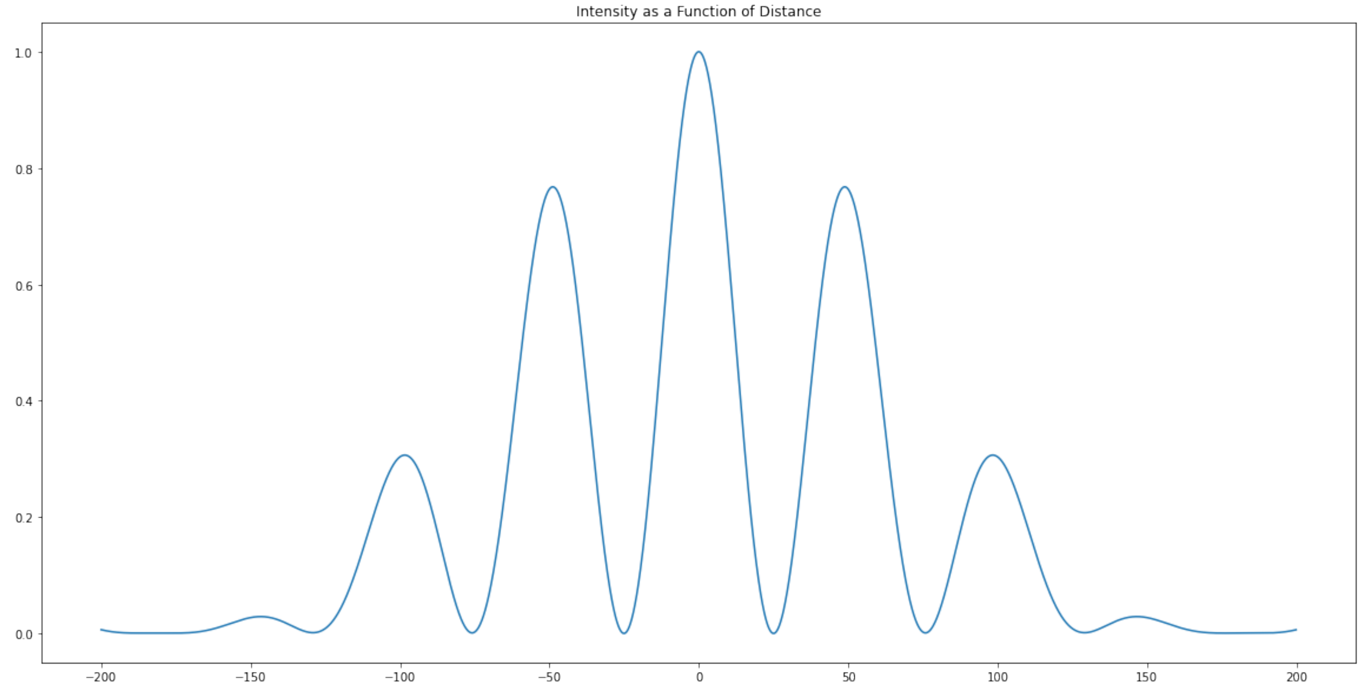
where $\beta = \dfrac{\pi w \sin\theta}{\lambda}$.

Problem 1.2

We will let $w = 3\lambda$ and $d = 10\lambda$, and let the distance between the slits and the wall be $D = 500\lambda$. This corresponds with setting the slits to be

```
slits = [
      [-5*wavelength, 3*wavelength, wavelength, wavelength, 1],
      [5*wavelength, 3*wavelength, wavelength, wavelength, 1]
    ]
```

and gives the following pattern:



The first minimum is at $(25.0 \pm 0.1)\lambda$. Note that the uncertainty comes from the fact that the for loop iterates through locations on the screen in intervals of 0.1λ.

Theoretically, the first minimum occurs when $\cos\left(\dfrac{\pi d}{\lambda}\theta\right) = 0$, which implies $\dfrac{\pi d\theta}{\lambda} = \dfrac{\pi}{2}$, which occurs when $\theta = \dfrac{\lambda}{2d}$. Using the approximation $\theta \approx \dfrac{x}{D}$, we get the first minimum to be:

$$x_{\min} = \frac{D}{2d}\lambda = 25\lambda. \tag{5}$$

However, the second minimum occurs at $(75.9 \pm 0.1)\lambda$, which disagrees with the theoretical $x_{\min,2} = 3x_{\min} = 75\lambda$. Even after accounting for non-small angles, i.e. $x_{3,\min} = D\arctan\left(\dfrac{3\lambda}{2d}\right) = 74.4\lambda$, it still doesn't agree. This is because the code doesn't adjust for thick slits properly.

Problem 1.3

For an angle θ, phasors of the two patterns will each have an amplitude $A_0(\theta)$, but are an angle $\frac{2\pi d}{\lambda}\theta$ apart. Therefore:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{6}$$

Note that we have already verified this experimentally using the code in the previous question. We just need to show that the interference pattern of a single (wide) slit is given by:

$$A(\theta) \propto \frac{\sin\beta}{\beta}, \tag{7}$$

which we can do similarly to what we did above by looking at the minimum and/or maximum points.

Problem 1.4

The separations are $d, 2d, \ldots, 2^{k-1}d$. Therefore, using the previous result recursively, we get:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right)\cos\left(\frac{2\pi d}{\lambda}\theta\right)\cdots\cos\left(\frac{2^{k-1}\pi d}{\lambda}\theta\right) \tag{8}$$

Let us consider the case where $d = w$ and take the limit $k \to \infty$ while keeping $2^k w$ constant. On the left hand side, we have a single slit of width $2^k d$, and we know the pattern is

$$A = \frac{\sin x}{x} \tag{9}$$

if we let $x = \frac{\pi(2^k d)\theta}{\lambda}$. On the right hand side, we have the same scenario as the previous part:

$$A = A_0(\theta)\cos(x/2)\cos(x/4)\cdots\cos\left(x/2^k\right).$$

As we take $k \to \infty$, we have $d \to 0$, implying A_0 goes to 1. Therefore, we get the desired relationship.

Part Two

Problem 2.1

Any aperture pattern can be represented by an aperture function, where a slit centered at x_0 that reduces the amplitude to f of the original can be represented by $f \cdot \delta(x - x_0)$ where $\delta(x)$ is the delta function.

It turns out that taking the Fourier transform of the aperture function can give the amplitude function. It may seem strange that we are transforming a variable with dimension of length x_1 (Which represents the location of the slits) to another dimension of length x_2 (which represents the location at the screen), when Fourier transforms typically take variables to their inverses (i.e. time to frequency). However, this is not the case. We are actually taking the transform from x_1 to $\frac{x_2}{\lambda D}$.

While we will not go into the details, it turns out taking the Fourier transform of the amplitude function will also get us the aperture function (i.e. transforming x_2 into x_1).

For a double slit, the intensity function is $\cos^2\left(\pi d\frac{x}{\lambda D}\right) = \cos^2(\pi d x_2)$. We can use the identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ to write:

$$\cos^2(\pi d \cdot x_2) = \frac{1}{2} + \frac{1}{2}\cos(2\pi d \cdot x_2). \tag{10}$$

Taking the Fourier Transform, we get:

$$\hat{f}\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi d\cdot x_2)\right]=\hat{f}\left[\frac{1}{2}\right]+\hat{f}\left[\frac{1}{2}\cos(2\pi d\cdot x_2)\right] \tag{11}$$

$$=\frac{1}{2}\hat{f}[1]+\frac{1}{2}\hat{f}[\cos(2\pi d\cdot x_2)] \tag{12}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{2}\left(\frac{\delta\left(x_1-\frac{2\pi d}{2\pi}\right)+\delta\left(x_1+\frac{2\pi d}{2\pi}\right)}{2}\right) \tag{13}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-d\right)+\frac{1}{4}\delta\left(x_1+d\right) \tag{14}$$

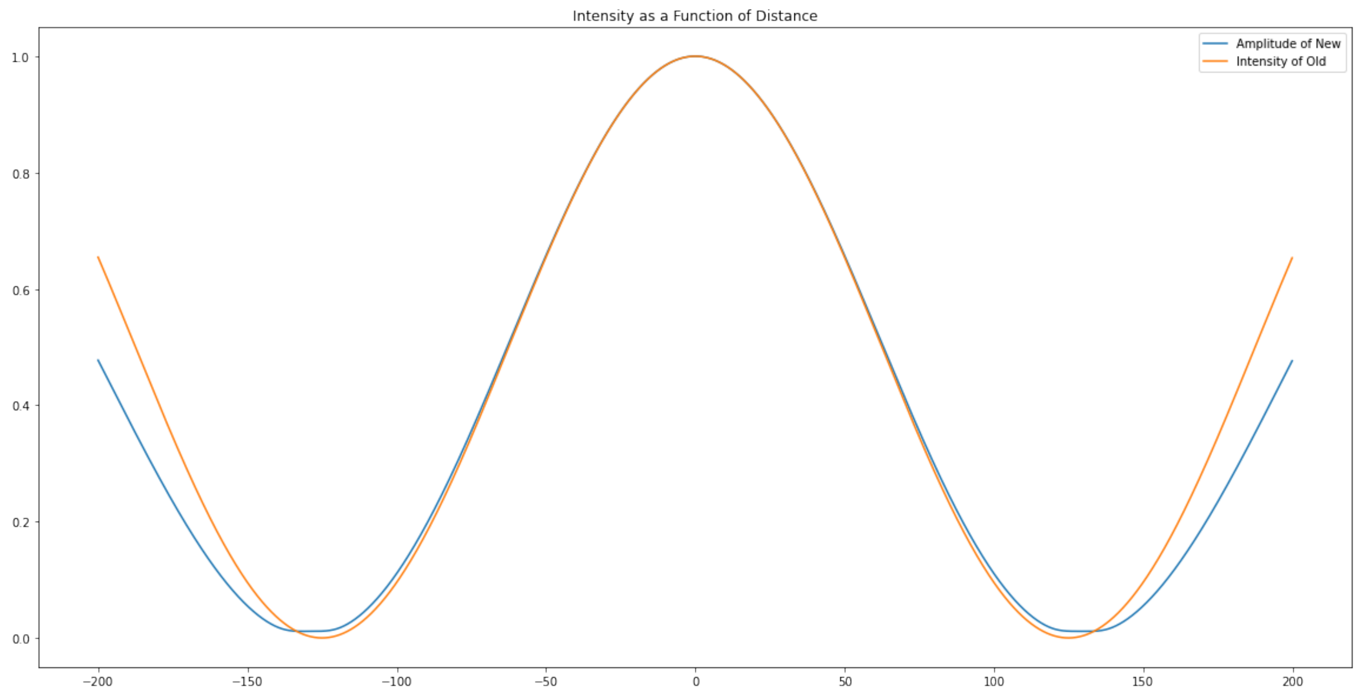
and to normalize it (since only the relative amplitudes matter), we're left with:

$$\delta(x_1)+\frac{1}{2}\delta(x_1-d)+\frac{1}{2}\delta(x_1+d). \tag{15}$$

We can verify this experimentally. Setting

```
slits = [
  [0*wavelength, 1*wavelength, wavelength, wavelength, 1],
  [-2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
  [2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
]
```

will give us the following plot, where we have plotted the intensity pattern of the previous aperture pattern for reference:



Notice that this is not perfect. This is because the Fourier Transform uses the far-field approximation, but the location of the first minimum isn't necessarily small enough.

Problem 2.2

We do a similar thing. Note that:

$$\cos^4(x)=\left(\frac{1}{2}(1+\cos(2x))\right)^2=\frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \tag{16}$$

Applying linearity of the fourier transform, we have

$$\begin{aligned}\hat{f}\left[\frac{1}{4}(1+2\cos(2ax)+\cos^2(2ax))\right] &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta\left(x-\frac{2a}{2\pi}\right)+\frac{1}{4}\delta\left(x+\frac{2a}{2\pi}\right)+\frac{1}{4}\hat{f}\left[\cos^2(2ax)\right] \\ &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta(x+d)+\frac{1}{4}\delta(x-d)+\frac{1}{4}\left(\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-2d\right)+\frac{1}{4}\delta\left(x_1+2d\right)\right) \\ &= \frac{3}{8}\delta(x)+\frac{1}{4}\delta(x\pm d)+\frac{1}{16}\delta(x\pm 2d)\end{aligned}$$

where we have let $a=\pi d$, and used the result from the previous problem. Normalizing this gives a slit in the middle that lets the full amplitude in, two slits located a distance d from the center that reduces the amplitude by $2/3$, and two slits located a distance $2d$ from the center that reduces the amplitude by $\frac{1}{6}$.

Problem 2.3

We want to find the Fourier transform:

$$\hat{f}[\cos^n(ax_2)] \tag{17}$$

However, multiplication in the x_2 domain is equivalent to convolution in the x_1 domain. We can apply the convolution theorem to say that:

$$\hat{f}[\cos^n(ax_2)] = \hat{f}[\cos(ax_2)]^n \tag{18}$$

where multiplication on the RHS is denoted by the convolution operator $\star$. We've seen that

$$\hat{f}[\cos(ax_2)] = \frac{1}{2}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right) \tag{19}$$

Note that convolution is associative and distributive. Using the fact that δ is the identity, we have the property:

$$\delta(x-a)\star\delta(x-b)=\delta(x-a-b) \tag{20}$$

Thus, we have:

$$\hat{f}[\cos(ax_2)]^n = \frac{1}{2^n}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right)^n$$

Note that $\frac{a}{2\pi}=\frac{d}{2}$. Expanding this using the distributive property, we see that each term contains k copies of $\delta(x-d/2)$ and $n-k$ copies of $\delta(x+d/2)$, which combines to give $\delta(x-kd/2+nd/2-kd/2)=\delta(x+(n/2-k)d)$ where $0\le k\le n$. If we label the $n+1$ slits as S_k, then S_k is located at a location of $(k-n/2)d$ and reduces the amplitude by $\frac{1}{2^n}\binom{n}{k}$. Here the $\binom{n}{k}$ comes in because there are $\binom{n}{k}$ terms in the convolution expansion that has k copies of the first term and $n-k$ copies of the second term (which corresponds to a unique slit).

Part Three

Problem 3.1

First, we make a few observations:

- The intensity pattern is symmetrical and centered at $x=0$: Therefore it probably consists of only cosines.
- The intensity never reaches 0, so the amplitude is always positive. We can then safely take the square root without worrying about reversibility issues.
- The number of peaks in between each period is 1, so it must mean that one frequency is exactly double the other. Looking at the period, we can conclude that two of the slits are located $\pm 5\lambda$ away from the center and $\pm 10\lambda$ frmo the center.

Using a data analysis tool such as Python or Excel, we find that the period is 102100.0λ.

Since the slits are at integer locations, we can write out a Fourier *series* by noting that each cosine would be in the form of:

$$\cos\left(d' \cdot \frac{2\pi}{\lambda D}x\right) \tag{21}$$

where d is an integer. Thus, let's clean up our data by:

- Taking the square root.
- Looking at only one period (to prevent edge effects from ruining the data)
- Plotting against $10\pi x/\lambda D$ instead of just x.

Taking a Fourier Series (i.e. by having $n = 2$), we get:

```
{y: a0 + a1*cos(w*x) + a2*cos(2*w*x) + b1*sin(w*x) + b2*sin(2*w*x)}

Parameter Value         Standard Deviation
a0        5.407222e-01 9.935952e-04
a1        1.419923e-01 1.439942e-03
a2        2.758724e-01 1.411563e-03
b1        -3.714775e-05 1.369781e-03
b2        -6.656304e-05 1.378570e-03
```

allowing us to reconstruct the sinusoidal wave. Taking a Fourier Transform, we get the aperture function. Here it is, for reference:

```
slits = [
      [0*wavelength, 1*wavelength, wavelength, wavelength, 3],
      [-5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [-10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
      [10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
    ]
```

Problem 3.2

Similar to the previous problem, we "cut" off the interference pattern at a small angle (but big enough such that the distinctive behaviour of the curve is captured). Scaling x to be $\frac{2\pi}{\lambda D}x$, we can determine the Fourier coefficients to be:

```
Parameter Value         Standard Deviation
a0        2.045300e-01 1.285129e-04
a1        3.294887e-01 1.439479e-04
a2        2.371716e-01 8.086478e-05
a3        1.652102e-01 8.350614e-05
a4        6.081714e-02 2.142034e-04
```

Taking the Fourier Transform analytically, we get

$$0.0854\delta(x \pm 4) + 0.2070\delta(x \pm 3) + 0.2973\delta(x \pm 2) + 0.4130\delta(x \pm 1) + 0.5126\delta(x) \tag{22}$$

Plotting f_X against x forms a straight line ($r^2 = 0.998$).

Problem 3.3

Note that the aperture function here is actually a triangle:

$$\text{tri}(x_1/5) \tag{23}$$

The Fourier Transform from x_1 to x_2 gives:

$$\mathrm{sinc}^2\left(\frac{5x_2}{\lambda D}\right). \tag{24}$$