2025 Online Physics Olympiad: Open Contest Problems

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- Use $g = 9.8$ m/s^2 in this contest, **unless otherwise specified**. See the constants sheet on the following page for other constants.
- This test contains 35 short answer questions. Each problem will have three possible attempts, unless otherwise specified.
- The weight of each question depends on our scoring system found here. Questions solved by fewer teams are worth more points, and the amount of points you get from a question decreases with the number of attempts that you take to solve it.
- Any team member is able to submit an attempt. Choosing to split up the work or doing each problem together is up to you. Note that after you have submitted an attempt, your teammates must refresh their page before they are able to see it.
- Answers should be within 1% relative accuracy unless otherwise specified.
- When submitting a response using scientific notation, please use exponential form. In other words, if your answer to a problem is $A \times 10^B$, please type AeB into the submission portal.
- A standard scientific or graphing handheld calculator *may* be used, along with technology and computer algebra systems like Wolfram Alpha or simple simulations. You are *allowed* to use Wikipedia or books in this exam.
- Asking for help on online forums or your teachers will be considered cheating and may result in a possible ban from future competitions. **The use of AI models such as ChatGPT is strictly forbidden.**
- Top scorers from this contest will qualify to compete in the Online Physics Olympiad *Invitational Contest*, which is an olympiad-style exam. More information will be provided to invitational qualifiers after the end of the *Open Contest*.
- In general, answer in SI units (meter, second, kilogram, watt, etc.) unless otherwise specified. Please input all angles in degrees unless otherwise specified.
- If the question asks to give your answer as a percent and your answer comes out to be "x%", please input the value x into the submission form.
- Do not put units in your answer on the submission portal! If your answer is "x meters", input only the value x into the submission portal.
- **Do not communicate information to anyone else except your team members before 12:00 PM UTC on August 11**.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}\ \text{mol}^{-1}$
- Universal gas constant, $R = 8.31$ J/(mol · K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\text{N} \cdot \text{m}^2)/\text{kg}^2$$
- Solar Mass
$$M_{\odot} = 1.988 \cdot 10^{30}\ \text{kg}$$
- Acceleration due to gravity, $g = 9.8\ \text{m/s}^2$
- 1 unified atomic mass unit,
$$1\ \text{u} = 1.66 \cdot 10^{-27}\ \text{kg} = 931\ \text{MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \text{J} \cdot \text{s} = 4.41 \cdot 10^{-15}\ \text{eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \text{C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\text{N} \cdot \text{m}^2)/\text{C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \text{T} \cdot \text{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$$
- 1 atmospheric pressure,
$$1\ \text{atm} = 1.01 \cdot 10^{5}\ \text{N/m}^2 = 1.01 \cdot 10^{5}\ \text{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}\ \text{m} \cdot \text{K}$
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2/\text{K}^4$$

Problems

1. Free Bird

The rest mass of the observable universe is 1.5×10^{53} kg as measured in the Earth frame. If βc is the maximum possible speed of an electron in this frame, find $1-\beta$. Ignore the effects of general relativity.

Solution 1:
The net momentum of the universe is orders of magnitude less than Mc. Thus, the electron is moving in one direction, and an equal momentum of photons moves in the opposite direction. Due to the relatively low mass of the electron in comparison to the mass of the universe, the electron has half the energy of the universe.

We then have

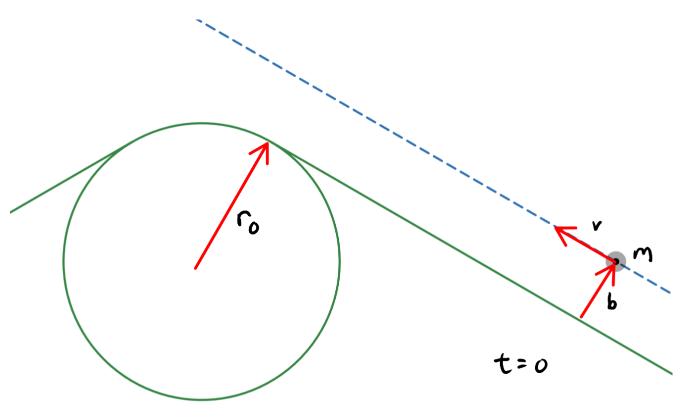
$$\gamma m_e c^2 = \frac{1}{2} M c^2$$

and thus

$$\begin{aligned} 1-\beta &\approx \frac{1}{2}(1-\beta^2) \\ &= \frac{1}{2}\left(\frac{2m_e}{M}\right)^2 \\ &= \boxed{7.377 \times 10^{-167}}. \end{aligned}$$

2. Racing Line

A race track is constructed from two semi-infinite straight sections, joined by a circular turn of radius $r_0 = 1$ m and angle $\theta = 120°$. A small race car of mass m moves along the straight section of the track at a constant speed v_0 and impact parameter $b = 0.5$ m from the inner wall of the track. At time $t = 0$ s, the driver rotates the steering wheel by a fixed amount, setting the car into uniform circular motion as it goes around the turn. At a later time $t = t_{min}$, the driver exits the turn by straightening the steering wheel and begins to move in a straight line staying distance b from the inner wall of the race track. Given $\mu_s = 1$ and assuming that the car does not skid or change its speed along its path, find the maximum speed possible for the car to complete the turn.



Solution 2:
Firstly, we are not allowed to collide with the track for obvious reasons. This imposes a maximum possible turn radius, given our initial and final distance of b from the walls of the track. We want the car to travel in this maximal radius turn because this allows it to carry the most speed without slipping, so

$$a_c = \frac{v^2}{r}.$$

Holding the radius of the turn constant, increasing a_c allows the car to travel at a greater v. However, there is a limit to how large a_c can be, since this acceleration is given by the static friction of the tires against the asphalt. The maximum centripetal acceleration possible is determined by the maximum grip force that the tires can provide:

$$ma_c = F_f \leq N\mu_s = mg\mu_s$$
$$a_c \leq g\mu_s$$
$$a_{c,max} = g\mu_s.$$

Therefore,

$$g\mu_s = \frac{v^2}{r}$$
$$v_{max} = \sqrt{\mu_s g r_{max}}.$$

All we need to do now is find r_{max}. After some geometric work, we find that

$$r_{max} = r_0 + b\left(\frac{1}{1-\cos(\theta/2)}\right).$$

Plugging in the values given, we find that $\boxed{v_{max} = 6.810\,\mathrm{m/s}}$.

3. The Cake is a Lie

Two identical rectangular portals are set horizontally in a watertight chamber such that no water can escape. A solid cylindrical water wheel with mass $m = 10$ kg and radius R is mounted such that its horizontal axle is halfway between the two portals. Each paddle of the wheel extends a small distance $r \ll R$ and the wheel is positioned so that when a paddle is horizontal it fits entirely between the portals and exactly lines up with the portal openings. Water entering a portal is instantaneously transported to the corresponding location on the other portal and emerges with the same velocity it had upon entry. The portals do not interact with the water wheel. At time $t = 0$, a horizontal slab of water with mass m, initially at rest, is released from directly under the top portal. Determine the energy of the wheel after 5929 minutes.

Solution 3:
5929 minutes is a long period of time, so we can make a crucial assumption: the time for the water to complete one cycle of moving through the portals will be much shorter than the total amount of

time, so we can approximate the speed ωR of the teeth to always equal the speed v of the water. Also, the system doesn't lose any energy to heat because for the vast majority of the time, the water is positioned above the teeth of the gear. The system gains energy at a rate mgv, so

$$\mathrm{d}E = \mathrm{d}W$$
$$\frac{3mv\mathrm{d}v}{2} = mgv\mathrm{d}t$$
$$v = \frac{2gt}{3}.$$

Then after 5929 minutes, the energy of the gear will be $mv^2/4 = \boxed{1.3504 \times 10^{13}\text{ J}}$.

4. Fiber Optics

A laser emitting light of wavelength in vacuum $\lambda = 1\ \mu$m is coupled with a straight optical fibre of refractive index $n = 1.5$, radius $r = 0.1$ mm and length $l = 100$ m. The laser periodically transmits impulses of very narrow width. Find the maximum frequency of pulse transmission that can guarantee that the receiver on the other end of the fibre can distinguish between pulses without interference.

Solution 4:
The pulse will spread by an amount of time equal to the difference between the minimum and maximum transit times. The minimum transit time is simply the straight line distance, $t_{min} = \frac{nl}{c} = 5 \times 10^{-7}$ s. The maximum transit time is the case where the light bounces off both sides of the cable at the critical angle, which is $\theta_c = 41.8°$. The distance travelled is therefore multiplied by an extra factor of $\frac{1}{\sin\theta_c} = 1.5$ and so $t_{max} - t_{min} = (1.5 - 1)t_{min} = 2.5 \times 10^{-7}$. Pulses must wait at least this long before they can be transmitted; therefore the maximum frequency of transmission is the reciprocal, i.e. $\boxed{4 \times 10^6\text{ Hz}}$.

5. Cosmological GPS

In a future where humanity is travelling outside of our galaxy, GPS satellites are scattered throughout Local Group and beyond to provide for the navigational needs of intergalactic travellers. A traveller in the Andromeda galaxy, 2.5 million light years away, connects to a satellite in the Milky Way. Calculate the absolute error in distance reported by the satellite caused by the expansion of the universe. The Hubble constant is $2.27 \times 10^{-18}\text{ s}^{-1}$, and it gives a cosmological velocity $v = H_0 d$ away from the observer.

Solution 5: The Hubble expansion gives a cosmological velocity $v = H_0 d$ away from the observer. Suppose a signal reaches distance r and time t since the emission of the signal. Then

$$\dot{r} = c - H_0 r, \tag{1}$$

so

$$r(t) = \frac{c}{H_0}(1 - e^{-H_0 t}). \tag{2}$$

The time taken to travel the distance $d = 2.5 \times 10^6$ ly is

$$t = -\frac{1}{H_0}\ln\left(1 - \frac{H_0 d}{c}\right) = 7.88 \times 10^{13}\text{ s}. \tag{3}$$

The time difference between travel time without cosmological expansion d/c and this time is

$$\Delta t = 7.06 \times 10^9\,\text{s}, \tag{4}$$

which corresponds to a distance error of

$$\Delta d = c\Delta t = \boxed{2.11 \times 10^{18}\,\text{m}}. \tag{5}$$

6. Water Droplet

Find the maximum height of a water droplet on a flat table. The density of water is $\rho = 1000$ kg/m^3 and the surface tension between water and air is $\gamma = 7.28 \cdot 10^{-2}$ N/m. Assume that the surface tensions between the table and air or water are 0.

Solution 6: Assume that the droplet's volume V is large enough that it is approximately flat with height h. Then, the surface energy is $E_\gamma \approx \pi r^2 \gamma = \frac{\gamma V}{h}$ and the gravitational energy is $E_g \approx \frac{1}{2} mgh = \frac{1}{2}\rho V g h$. The total energy $E_\gamma + E_g$ is minimized for $\boxed{h = \sqrt{\frac{2\gamma}{\rho g}}} \approx 3.85$ mm. In reality, the water droplet's height will asymptotically approach this value as V increases.

7. A Balancing Act

A pencil of uniform density and length $\ell = 10$ cm is vertically oriented and a frictionless pivot is put a distance d below the midpoint of the pencil. The pencil is then let go. Find the maximum d such that the pencil will remain stably upright. Neglect all forces except gravitational forces and the normal force from the pivot. Treat the Earth as spherically symmetric with radius 6378 km.

Solution 7: We need to consider the net torque on the pencil from tidal forces. Letting the acceleration at the pivot be $-g_0\hat{y}$, the acceleration at $\langle x, y\rangle$ is

$$\vec{g} = -g_0 \frac{R^2}{(R+y)^2 + x^2}\hat{r} \approx -g_0 \left(1 - \frac{2y}{R}\right)\left\langle -\frac{x}{R}, -1\right\rangle \approx g_0 \left\langle -\frac{x}{R}, -1 + \frac{2y}{R}\right\rangle$$

Let the pencil be displaced by angle θ. At distance a from the pivot, we have $\vec{r} \approx \langle -a\theta, a\rangle$, so the torque per unit length is:

$$d\tau = \vec{r} \times \lambda \vec{g}\, da \approx \langle -a\theta, a\rangle \times \lambda g_0 \left\langle \frac{a\theta}{R}, -1 + \frac{2a}{R}\right\rangle da = \theta\lambda g_0 \left(a - \frac{3a^2}{R}\right)\hat{z}\, da$$

$$\begin{aligned}
\Rightarrow \tau_z = \int d\tau_z &= \int_{-\ell/2+d}^{\ell/2+d} \theta\lambda g_0 \left(a - \frac{3a^2}{R}\right) da \\
&= \frac{\theta\lambda g_0}{2}\left(\left(\frac{\ell}{2}+d\right)^2 - \left(-\frac{\ell}{2}+d\right)^2\right) - \frac{\theta\lambda g_0}{R}\left(\left(\frac{\ell}{2}+d\right)^3 - \left(-\frac{\ell}{2}+d\right)^3\right) \\
&\approx \theta\lambda g_0 \left(d\ell - \frac{\ell^3}{4R}\right)
\end{aligned}$$

For this to be a restoring torque, we need $\boxed{d < \frac{\ell^2}{4R}} = 3.92 \cdot 10^{-10}\,\mathrm{m}$.

The following applies for the next two problems. It is possible to do a cool trick using a plastic water bottle. In the following two problems, we will analyze the conditions required for such a trick to be performed. The procedure is as follows:

- Empty a small plastic water bottle, and dry it such that it has no liquid water inside. The interior air will have the same temperature and composition as the room's air.
- Screw the cap tightly onto the bottle such that it is airtight.
- Twist the bottle, such that the air in the bottle is effectively compressed to 95% of its original volume. The twisting process is done quickly enough that negligible heat is transferred to/from the interior air.
- After waiting for a while, the interior air returns to room temperature. Unscrew the bottle cap, causing it to pop out quickly. A fine mist is produced.

You may treat the air, and all of its components, as an ideal diatomic gas. The room temperature is $T_0 = 20.0°\mathrm{C}$. If you need the vapour pressure of water, use the Buck formula at this online calculator.

8. Bottle trick 1

Find the minimum relative humidity the room's air needs to have for this trick to work. Give your answer as a decimal (where 1 would mean that the air is saturated with water vapour).

Solution 8: First, the gas undergoes isothermal compression.

$$P_1 V_1 = P_0 V_0$$

Then, it undergoes adiabatic expansion until the gas reaches atmospheric pressure:

$$P_0 V_2^\gamma = P_1 V_1^\gamma \implies P_0^{1-\gamma} T_2^\gamma = P_1^{1-\gamma} T_1^\gamma$$

The final temperature of the gas after expansion is:

$$T_2 = T_0 \left(\frac{V_1}{V_0}\right)^{(\gamma-1)/\gamma} = 15.735°\mathrm{C}.$$

For a mist to be produced, the partial pressure of water vapor has to exceed the saturation pressure. Since the partial pressure of the water vapor is the same as in the original air, we just need the temperature to be low enough such that the saturation pressure at this lower temperature is equal to the partial pressure:

$$\frac{P_{\mathrm{vap}}(15.735°\mathrm{C})}{P_{\mathrm{vap}}(20.0°\mathrm{C})} = 0.76447.$$

9. Bottle trick 2

Immediately after twisting the bottle, water droplets form on the inside wall of the water bottle. Find the minimum relative humidity of the air that allows for the water droplets to form.

Solution 9:
Adiabatic compression:

$$\frac{P_1}{P_0} = \left(\frac{V_1}{V_0}\right)^{-\gamma} = 1.0745.$$

The partial pressure of the water vapour is also scaled by this amount, as we assume that all gases are diatomic. It condenses when it exceeds the vapour pressure at the original temperature so that the water vapor can condense on the walls. Hence, the minimum relative humidity is

$$\frac{P_0}{P_1} = \left(\frac{V_1}{V_0}\right)^{\gamma} = 0.95^{7/5} = 0.9307.$$

Since this is higher than the condition required for the mist to form, even if no water droplets are observed on the walls, it is still possible for the trick to be done. This is why only some videos on the internet show water droplets forming on the bottle. Also, since this condition is weaker than the first condition, we can simply ignore the condensation of water when finding the lower bound of relative humidity in the first question.

10. Frosty

Four ice cubes are placed in different environments.All the ice cubes are the same size. Your goal is to determine which of ice cube 1 or 2 fully turns to liquid first, and which of ice cube 3 or 4 fully turns to vapor first.

1. Ice cube 1 is at room temperature in a slightly **more** humid atmosphere.
2. Ice cube 2 is at room temperature in a slightly **less** humid atmosphere.
3. Ice cube 3 is on a hot plate at 100°C in a slightly **more** humid atmosphere.
4. Ice cube 4 is on a hot plate at 100°C in a slightly **less** humid atmosphere.

You may neglect heat conduction and heat capacity of air, and assume that the difference in humidity does not significantly change the boiling point. The room is much larger than the ice cube and the ice cubes are of similar size to the hot plate.

Submit the solution as a 2 digit number as follows:

- The tens digit is 1 if ice cube 1 melts first, 2 if ice cube 2 melts first and 9 if they take the same time.
- The ones digit is 3 if ice cube 3 boils first, 4 if ice cube 4 boils first and 9 if they take the same time.

For example, if ice cube 2 melts first and ice cube 4 boils first, submit the answer 24. **Only one attempt will be accepted for this problem.**

Solution 10:
Melting: The low humidity ice cube will vaporise as it melts, taking heat away from the cube and therefore it will slow down the process of melting. Hence the high humidity ice cube melts first.

Boiling: The final state for both cubes is the exact same, bar a slightly lower vapor pressure in the case of low humidity. Hence the heat required to boil the entire cube must be the same for both. For the cube in low humidity, some of the water must vaporise as it heats up, reducing the temperature of the cube and hence drawing a greater amount of power from the hot plate through conduction. This means it receives heat faster through the hot plate and therefore will boil faster.

The answer is therefore $\boxed{14}$.

11. Electric Disco

A thin, conductive disk of radius $R = 1\,\mathrm{m}$ is centred at the origin of the xy-plane, with axis perpendicular to the plane. The potential at the boundary is given by $V(x, y) = x(xy + 1) + y(y + 1)$ V, where x and y are in meters. What angle, in radians and measured in the counterclockwise direction, does the current density at the origin make with respect to the positive x-axis?

Solution 11: Consider the decomposition of $V(R, \theta)$ into Fourier components:

$$V(R,\theta) = A_0 + \sum_{i=1}^{\infty} (A_i \cos(i\theta) + B_i \sin(i\theta))$$

The constant component and those with $i \geq 2$ have zero electric field at the center by symmetry. On the other hand, $E = E_0\hat{x}$ gives potential $V(R, \theta) = -E_0 R\cos(\theta)$. Thus, the electric field at the center is $\vec{E}_0 = \langle -A_1/R, -B_1/R \rangle$ by linearity.

For the given potential, we have:

$$\begin{aligned} V(R,\theta) &= \cos(\theta)(\sin(\theta)\cos(\theta) + 1) + \sin(\theta)(\sin(\theta) + 1) \\ &= \frac{\cos(\theta)\sin(2\theta)}{2} + \cos(\theta) + \frac{1 - \cos(2\theta)}{2} + \sin(\theta) \\ &= \frac{1}{2} + \cos(\theta) + \frac{5}{4}\sin(\theta) - \frac{\cos(2\theta)}{2} + \frac{\sin(3\theta)}{4} \end{aligned}$$

Thus, the electric field points in the direction $\langle -1, -5/4 \rangle$, corresponding to an angle of $\boxed{4.037\,\mathrm{rad}}$. The coefficients A_1 and B_1 can also be extracted using Fourier's trick.

12. Bungee Jumping

Consider a thin hoop of radius 1 m, upon which lies four frictionless beads of mass 0.1 kg forming two pairs of beads connected by centrally pivoting diametric rods, not unlike a pair of scissors. Consider four elastic "bungee" cords, created by cutting up a single elastic cord of relaxed length 5 m and spring constant

10 N/m. These four elastic cords, connected bead to bead, are relaxed but not slack at equilibrium. Find the period of small oscillations about this equilibrium point.

Solution 12:
The beads form a rectangle. Let ℓ_1, ℓ_2 represent the sides of the rectangle. We have the system

$$\ell_1^2 + \ell_2^2 = 2r$$
$$2(\ell_1 + \ell_2) = L.$$

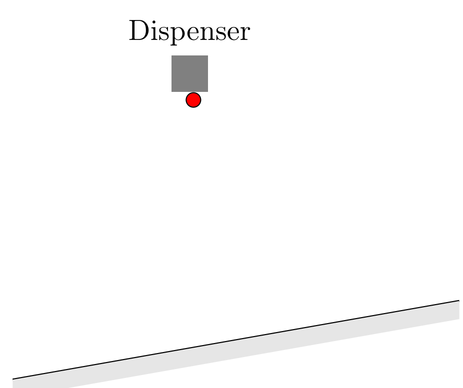
Solving, we see that $\ell_{1,2} = \dfrac{L \pm \sqrt{32r^2 - L^2}}{4} = 0.589, 1.911\,\text{m}$.
It's important to note that bungees only act in tension. Thus, if our bead is displaced on the positive (clockwise) side of the equilibrium point, only the shorter bungee acts and vice versa. Thus we must add the half-periods of simple harmonic motion of with two different spring constants.

From here, it's easiest to work with energy. Consider a half-period where ℓ_1 is stretched. If a bead moves by distance x from its equilibrium position, the kinetic energy is $K = 2m\dot{x}^2$ while the change in width is $\Delta\ell_1 = (\ell_2/r)x$. Thus, because the spring constant is $k_1 = k\dfrac{L}{\ell_1}$, we have $U = 2 \cdot \dfrac{1}{2}k_1 \Delta\ell_1^2 = \dfrac{kL\ell_2^2}{\ell_1 r^2}x^2$. The half-period is then $T_1 = \pi\sqrt{\dfrac{2m\ell_1 r^2}{kL\ell_2^2}}$. This gives:

$$T = T_1 + T_2 = \boxed{\pi\sqrt{\frac{2mr^2}{kL}}\left(\sqrt{\frac{\ell_2}{\ell_1^2}} + \sqrt{\frac{\ell_1}{\ell_2^2}}\right)} = 0.546\,\text{s}$$

13. Bouncing Down the Slope

A dispenser releases small identical uniform disks of mass $m = 2$ kg from rest at a height $h = 50$ m above a long frictionless wedge of mass $M = 500$ kg at a uniform rate of 50 disks per second, beginning at $t = 0$. The wedge is angled at $\theta = 10^\circ$ above the horizontal and is fixed on top of a horizontal scale. The disks collide with the slope with a coefficient of restitution $\alpha = 0.95$. Find the reading on the scale after time $t = 150\,\text{s}$ in Newtons, averaged over the timescale of a few ball collisions (the scale measures the vertical force exerted on it). Assume that the incline is sufficiently long such that no disks leave the incline.



Solution 13:
Sol 1
We will analyze the average impulse delivered to the slope by a single disk. Rotate the coordinate system so that the x-axis is aligned parallel to the slope, and the y-axis is aligned perpendicular to it. The speed of the ball upon its first collision with the slope is $\sqrt{2gh}$. The y-component of this velocity is

$$v_0 = \sqrt{2gh}\cos\theta.$$

Let v_n denote the magnitude of the y-component of the velocity immediately after the nth collision with the slope. We can see that

$$\begin{aligned} v_1 &= \alpha\sqrt{2gh}\cos\theta \\ v_2 &= \alpha^2\sqrt{2gh}\cos\theta \\ &\dots \\ v_n &= \alpha^n\sqrt{2gh}\cos\theta. \end{aligned}$$

The time of flight after the nth collision is

$$t_n = \frac{2v_n}{g\cos\theta} = \frac{2\alpha^n\sqrt{2gh}}{g}.$$

Thus, we can see that the total time of flight for each ball is

$$t_{\text{total}} = \frac{\frac{2\alpha\sqrt{2gh}}{g}}{1-\alpha} + \sqrt{\frac{2h}{g}} = \sqrt{\frac{2h}{g}\frac{1+\alpha}{1-\alpha}}.$$

The number of disks that have completed their bouncing process and are sliding down the slope is

$$k(t - t_{\text{total}}).$$

Note that since the rate of disks released is k balls/second, the number of collisions at each collision point per second is k. Thus, the average force exerted on the slope from the disks is

$$\begin{aligned} F_{\text{balls}} &= k(t - t_{\text{total}})(mg\cos\theta) + k(mv_1 + mv_0 + mv_2 + mv_1 + mv_3 + mv_2 + \dots) \\ &= k(t - t_{\text{total}})(mg\cos\theta) + kmv_0 + 2km\frac{\alpha\sqrt{2gh}\cos\theta}{1-\alpha}. \end{aligned}$$

Finally,

$$\begin{aligned} F &= F_{\text{balls}}\cos\theta + Mg \\ &= \left(k\left(t - \sqrt{\frac{2h}{g}\frac{1+\alpha}{1-\alpha}}\right)(mg\cos\theta) + km\sqrt{2gh}\cos\theta + 2km\frac{\alpha\sqrt{2gh}\cos\theta}{1-\alpha}\right)\cos\theta + Mg. \end{aligned}$$

Thus we obtain $\boxed{F = 1.475\cdot 10^5\,\text{N}.}$

Sol 2
Note: in this solution, we refer to "vertical" and "horizontal" in the rotated reference frame where the inclined plane is horizontal.
Follow Solution 1 until the total time of flight, which we calculate to be

$$t_{\text{total}} = \sqrt{\frac{2h}{g}\frac{1+\alpha}{1-\alpha}} = 125 \text{ s}$$

This is less than $t = 150$ s. This means that at $t = 150$ s, some of the disks will just be sliding down the slope.
Consider the positions of all disks that are in the air at some time when a ball is being released from the dispenser, and after a time $1/k$. Every disk moves into the position of another disk, except for the first disk that was released. Since the first disk has zero vertical velocity and the last disk (the one that was just dispensed) has zero vertical velocity, the total momentum of all the disks in the air does not change. Hence, the momentum is a constant $hmk \cos\theta$ downwards. Now, using Newton's 2nd law, the normal force from the inclined plane must be just enough to balance the vertical component of gravitational force on the disks in the air, which is $mgkt \cos\theta$. This is the normal force of the inclined plane on the disks. It's component normal to the scale contributes an additional normal force of the scale on the inclined plane. Then the total normal force is

$$N_{tot} = Mg + mgkt\cos^2\theta = \boxed{1.475 \cdot 10^5 \text{ N}}$$

14. Gravity Cable Car

An inextensible cable of length l is hung over a valley of width $l_0 = l/2$. The two walls of the valley are at the same altitude. A cable car is fastened to rollers at one end of the rope, gently lowered along the wall of the valley until it cannot go any further down, and let go. If the cable car takes time T to cross the valley, find $T\sqrt{g/\ell_0}$. You may assume the mass of the cable car is much larger than the mass of the cable itself, and neglect friction.

Solution 14: The cable car will trace out an ellipse with focal points at (0,0) and $(0, l_0)$. The semi-major axis of the ellipse is $a = l/2 = l_0$. The focal length of the ellipse is $l_0/2$. Hence the semi-minor axis of the ellipse is $b = \sqrt{a^2 - f^2} = \sqrt{l^2 - l_0^2}/2 = \frac{\sqrt{3}}{2}l_0$.

We have $\frac{(x-l_0/2)^2}{(l/2)^2} + \frac{y^2}{(l_0\sqrt{3}/2)^2} = 1$. Conservation of energy gives

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy = -gy_0 = -\frac{3gl_0}{4}. \tag{6}$$

We have

$$\dot{x}\left(x - \frac{l_0}{2}\right) + \frac{4}{3}\dot{y}y = 0. \tag{7}$$

Substituting into (7), we get

$$\frac{1}{2}\dot{x}^2\left(1 + \frac{9}{16y^2}\left(x - \frac{l_0}{2}\right)^2\right) = g\left(y - \frac{3l_0}{4}\right) \tag{8}$$

$$\frac{1}{2}\dot{x}^2\frac{l_0^2 - \frac{1}{4}(x - l_0/2)^2}{l_0^2 - (x - l_0/2)^2} = g\left(\frac{\sqrt{3}}{2}\sqrt{l_0^2 - \left(x - \frac{l_0}{2}\right)^2} - \frac{3l_0}{4}\right). \tag{9}$$

Substitute $x' = x - l_0/2$:

$$\frac{1}{2}\dot{x}^2\frac{l_0^2 - \frac{1}{4}x'^2}{l_0^2 - x'^2} = g\left(\frac{\sqrt{3}}{2}\sqrt{l_0^2 - x'^2} - \frac{3l_0}{4}\right). \tag{10}$$

$$T\sqrt{\frac{g}{l_0}} = \int_{-\frac{l_0}{2}}^{\frac{l_0}{2}} \sqrt{\frac{1-\frac{1}{4}(\frac{x}{l_0})^2}{1-(\frac{x}{l_0})^2}} \frac{1}{\sqrt{\sqrt{3}\sqrt{1-(\frac{x}{l_0})^2}-\frac{3}{2}}}\,dx \tag{11}$$

The RHS of (11) evaluates to 3.38. Therefore the time taken is

$$T = \boxed{3.38\sqrt{\frac{l_0}{g}}}. \tag{12}$$

15. Magnetic Monopole

Consider a hypothetical magnetic monopole of magnetic charge $q_{\mathrm{m}} = 10^{-5}$ T·m^2 and mass $m = 10^{-10}$ kg placed at the center of a fixed ring of resistance $R = 5\ \Omega$ and radius $r = 2$ m. The monopole is given an initial velocity along the axis of the ring. Find the minimum velocity v_{m} such that the monopole can escape from the ring. Assume that the magnetic field produced due to a magnetic monopole is $\vec{B} = \frac{q_{\mathrm{m}}}{4\pi r^2}\hat{r}$ and that the magnetic force acting on a magnetic monopole is $\vec{F} = \frac{q_{\mathrm{m}}}{\mu_0}\vec{B}$.

Solution 15:

First we find the flux through the loop due to the monopole when it is a distance h away from the center of the ring,

$$\Phi_B = \frac{1}{4\pi}\int_0^r \frac{q_{\mathrm{m}}}{x^2+h^2}\frac{h}{\sqrt{x^2+h^2}}2\pi x\mathrm{d}x = \frac{1}{2}q_{\mathrm{m}}\left(1-\frac{h}{\sqrt{r^2+h^2}}\right).$$

Now we take the time derivative of the flux to find the emf and divide the result by the resistance to find the induced current in the loop. Let I be the current and v be the velocity:

$$I = \left|\frac{d\Phi_B}{dt}\right|\frac{1}{R} = \frac{q_{\mathrm{m}}vr^2}{2R(r^2+h^2)^{3/2}}.$$

By Biot-Savart's law, the magnetic field due to the current loop at the location of the monopole is

$$B = \frac{\mu_0 I r^2}{2(h^2+r^2)^{\frac{3}{2}}} = \frac{\mu_0 r^4 v q_{\mathrm{m}}}{4R(r^2+h^2)^3}.$$

Thus,

$$F = \frac{q_{\mathrm{m}}B}{\mu_0} = \frac{r^4 v q_{\mathrm{m}}^2}{4R(r^2+h^2)^3} = -mv\frac{\mathrm{d}v}{\mathrm{d}h}$$

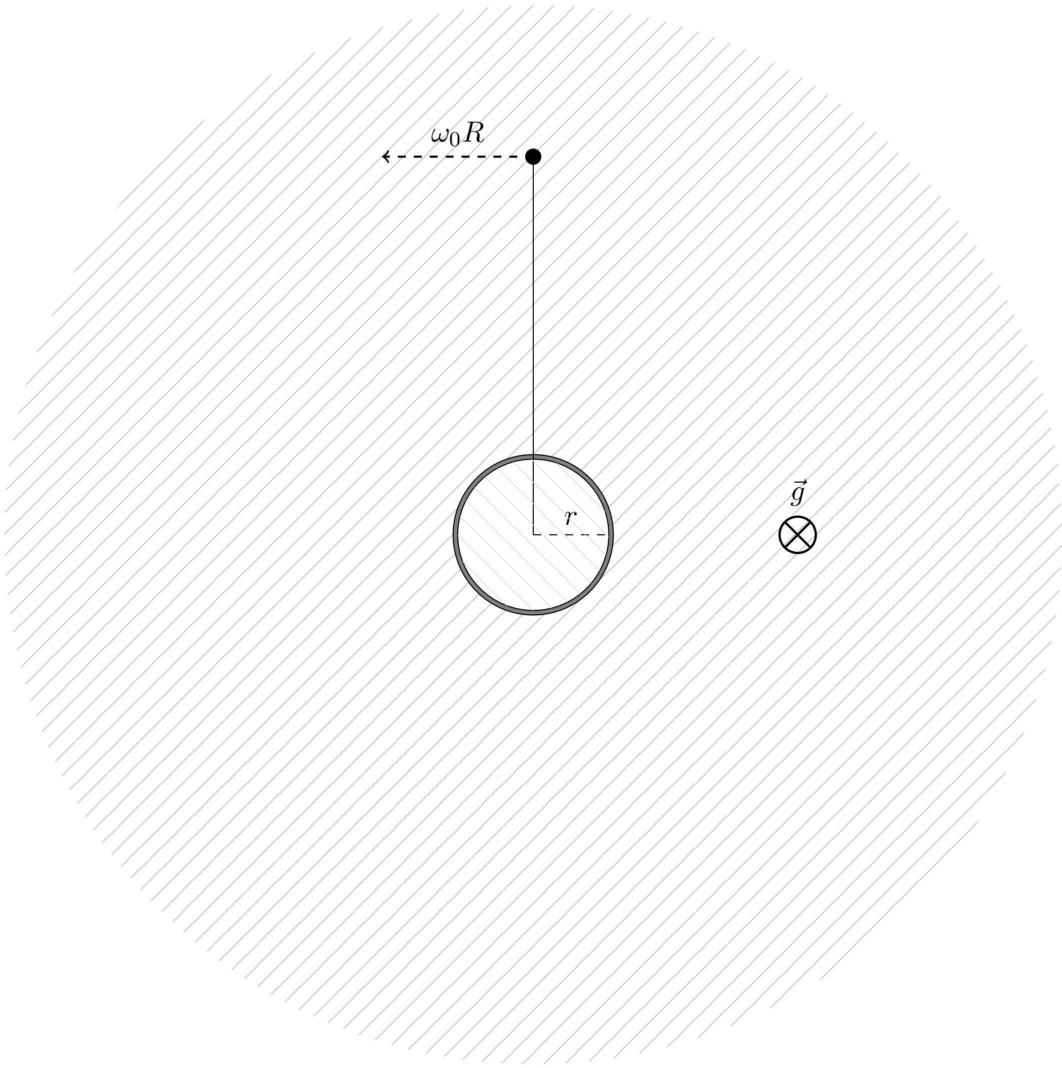
$$-\int_{v_{\mathrm{m}}}^0 \mathrm{d}v = \int_0^\infty \frac{r^4 q_{\mathrm{m}}^2}{4mR(r^2+h^2)^3}\mathrm{d}h.$$

After using a trig substitution to evaluate the integral, we arrive at the final answer:

$$v_{\mathrm{m}} = \frac{3\pi q_{\mathrm{m}}^2}{64Rrm} \approx \boxed{1.47\cdot 10^{-2}\,\mathrm{m/s}}.$$

16. Variant Roulette, House Edge

A spinner, modeled by a massless rod of length R connected to a point mass, is attached to the center of a massless cylinder (pin) of radius r. The cylinder is almost perfectly fit in a hole in a floor, that is, the cylinder is slightly smaller than the hole. The spinner, laid perfectly flat on the floor and starting at $\theta = 0$, is given a random angular velocity in the interval $(0, \omega_L)$, where ω_L produces exactly $N \gg 1$ full rotations (assume classical mechanics). What is the probability that it lands in the region $\pi/2 < \theta < 3\pi/2$ if there is a coefficient of friction $\mu = 50$ between the pin and the hole and $r/R = 0.2$?



Solution 16:
We will calculate the torque on the cylinder-spinner apparatus. There are 3 frictional forces: one between the rod and the floor, one between the cylinder and the floor, and one between the cylinder and the wall. We label the former two torques as f—which by dimensional analysis, will just drop out—and find the latter to be equal to $\mu m\omega^2 rR$. Then,

$$I\alpha = -f - \mu m\omega^2 rR$$
$$I\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = -f - \mu m\omega^2 rR.$$

Solving this differential equation for ω in terms of θ,

$$\omega = \sqrt{\frac{f}{\mu m r R}(e^{2\mu\theta r/R} - 1)}.$$

The probability that the spinner lands in the desired region is given by the sum of all angular velocities corresponding with a θ in that region divided by our total range of angular velocities. Letting $\lambda = 2\pi\mu r/R$, every 2π,

$$\begin{aligned}\Delta\omega_n &= \sqrt{\frac{f}{\mu m r R}(e^{2\lambda(n+3/4)} - 1)} - \sqrt{\frac{f}{\mu m r R}(e^{2\lambda(n+1/4)} - 1)} \\ &\approx \sqrt{\frac{f}{\mu m r R}}(e^{\lambda(n+3/4)} - e^{\lambda(n+1/4)}),\end{aligned}$$

since $e^{2\lambda n} \gg 1$ is always true. Summing using the formula for geometric series, dividing by our total range of angular velocities, and taking the limit as $N \to \infty$,

$$\begin{aligned}p &= \lim_{N\to\infty} \frac{\sum_{n=0}^{N}(e^{\lambda(n+3/4)} - e^{\lambda(n+1/4)})}{e^{\lambda(N+1)} - 1} \\ &= \lim_{N\to\infty} \frac{e^{3\lambda/4} - e^{\lambda/4}}{e^{\lambda(N+1)} - 1} \frac{e^{\lambda(N+1)} - 1}{e^{\lambda} - 1} \\ &= \frac{e^{3\lambda/4} - e^{\lambda/4}}{e^{\lambda} - 1}.\end{aligned}$$

Thus, our answer[a] is $\boxed{p = 1.507 \times 10^{-7}}$.

[a] At first glance, this answer obviously seems questionable; you may have had to convince yourself that it is. The reason for this extremely low probability is that ω exponentially scales with θ and depends on the friction factor, which in this case was chosen to be unreasonably—and unphysically—high.

17. Squid Game

We model a prehensile octopus tentacle of length $L = 2$ m as made of very many servomotors, each of which can apply a torque $\tau = 4$ N·m about any axis. An octopus wishes to pick up a cylinder of radius $r = 4$ cm and mass m by wrapping the tentacle around the side and lifting it such that its axis points straight up. What is the maximum M the octopus can pick up? Coefficient of friction between tentacle and cylinder is $\mu = 0.001$. Neglect buoyancy. Respond with 42.0 if the answer is infinite.

Hint: μ is very small, so the octopus is able to lift the cylinder if and only if it can get a grip on it.

Solution 17: Let's do virtual work on the cylinder. We note that the energy associated with the tentacle is equal to (summing across a large number of motors)

$$\sum_i \tau(\theta_{i+1} - \theta_i) = \tau\Delta(\theta_{\text{end}} - \theta_0) = \tau\frac{L}{r}$$

For a displacement of δr, therefore,

$$N\delta r = \frac{\tau L}{r^2}\delta r \to N = \frac{\tau L}{r^2} \to mg = F_\mu = \frac{\mu \tau L}{r^2}$$

and thus

$$m = \boxed{\frac{\mu \tau L}{g r^2}} = 0.510 \text{ kg}$$

Note: Since μ is small, the fact that the small shrinking of the radius results in the tentacle "slipping" across the surface of the cylinder is irrelevant.

18. Radioactive

A thin disk made up of $n = 1.00 \cdot 10^{-3}$ mol of ${}^{210}_{84}$Po lies face down on a frictionless table. Find the expected speed of the disk after one half life. ${}^{210}_{84}$Po has mass $M_0 = 195.59781$ GeV$/c^2$ and decays to ${}^{206}_{82}$Pb with mass $M_1 = 191.86400$ GeV$/c^2$, emitting an alpha particle with mass $m_\alpha = 3.72738$ GeV$/c^2$ and two electrons with negligible kinetic energy.

Solution 18: First, we'll find the momentum p of the alpha particle. The total kinetic energy of the lead/alpha system is $K = (M_0 - M_1 - m_\alpha - 2m_e)c^2 = 5.407\,\text{MeV}$, which is sufficiently small to be nonrelativistic. Then, we need:

$$\frac{p^2}{2M_1} + \frac{p^2}{2m_\alpha} = K \Longrightarrow p = \sqrt{2K\frac{M_1 m_\alpha}{M_1 + m_\alpha}} = 1.062 \cdot 10^{-19}\,\text{N}\cdot\text{s}$$

It's easiest to proceed in momentum space because we don't need to take into account the change in the disk's mass. In a single emission, the components of the momentum satisfy $p_x^2 + p_y^2 + p_z^2 = p^2$. Thus, each $\langle p_i^2 \rangle = \frac{p^2}{3}$ by symmetry.

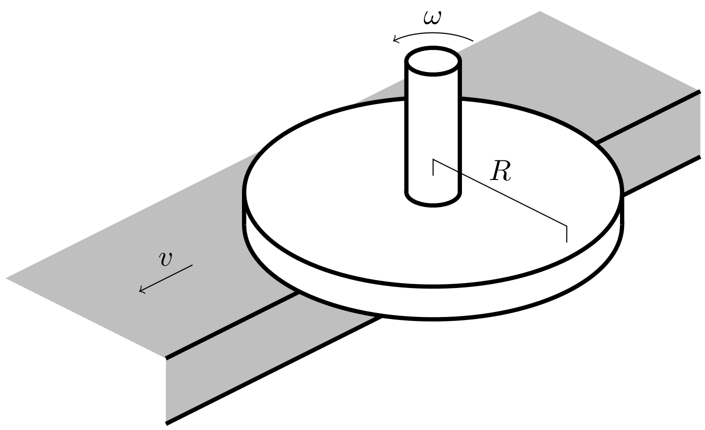
Because each emission is independent, the momenta P_x and P_y of the disk after the whole process have variance $\sigma^2 = \frac{Np^2}{3} = \frac{N_a n p^2}{6}$, where N is the number of decays (note that $P_z = 0$). Furthermore, P_x and P_y have mean 0 and are normally distributed by the central limit theorem. Thus, $\langle P \rangle$ is the expected value of a chi distribution with 2 degrees of freedom, giving $\langle P \rangle = \sigma\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi N_a n p^2}{12}} = 1.333 \cdot 10^{-9}\,\text{N}\cdot\text{s}$. This can also be found through integration.

Finally, the mass of the disk after the decays is $M = N_a n \frac{M_0 + M_1}{2} = 2.080 \cdot 10^{-4}\,\text{kg}$. This gives:

$$\langle v \rangle = \frac{\langle P \rangle}{M} = \boxed{6.41 \cdot 10^{-6}\,\frac{\text{m}}{\text{s}}}$$

19. Arctic Circle

A freely rotating disk of radius R, made of ice at its melting point, is pressed against a belt moving at speed v such that the disk's center is at the edge of the belt. Due to drag, the disk rotates at constant angular velocity. The drag force per unit area is linear (proportional to relative velocity). Find the ratio between the largest and smallest time-averaged rates of melting at a point on the disk. Assume that the pressure is constant across the disk.



Solution 19: First, we'll find the angular velocity ω of the disk. Placing the belt in the $x > 0$ plane, the belt's velocity is $\langle 0, v\rangle$ while the disk's velocity is $\omega\hat{z}\times\vec{r} = \langle -\omega y, \omega x\rangle$. Thus, the relative velocity is $\vec{v}_r = \langle \omega y, v - \omega x\rangle = \omega\langle y, a - x\rangle$ where $a = v/\omega$. Then, we have

$$\mathrm{d}\vec{\tau} \propto \vec{r}\times\vec{v}_r\ \mathrm{d}A \propto (ax - x^2 - y^2)\hat{z}\ \mathrm{d}A = (ar\cos(\theta) - r^2)\hat{z}\ \mathrm{d}A$$

$$\implies \vec{\tau} = \iint \mathrm{d}\vec{\tau} \propto \int_0^R \int_{-\pi/2}^{\pi/2} (ar\cos(\theta) - r^2)\hat{z}\cdot r\ \mathrm{d}r\ \mathrm{d}\theta = \left(\frac{2aR^3}{3} - \frac{\pi R^4}{4}\right)\hat{z}.$$

For the disk to rotate at constant angular velocity, we must have $\vec{\tau} = \vec{0}$, giving $a = 3\pi R/8$. Interestingly, no point on the disk is moving as fast as the belt!

Now, we'll consider the melting of the ice. Due to latent heat, the melting rate is proportional to the power dissipated per unit area. This is in turn proportional to v_r^2 since $P = Fv_r$ and $F \propto v_r$. Then, since a point on the disk at radius r contacts a semicircle from $\theta = -\pi/2$ to $\pi/2$, the average melting rate at radius r is proportional to:

$$\int_{-\pi/2}^{\pi/2} v_r^2\ d\theta = \int_{-\pi/2}^{\pi/2} y^2 + (a-x)^2\ d\theta = \int_{-\pi/2}^{\pi/2} r^2 - 2ar\cos(\theta) + a^2\ d\theta = \pi r^2 - \frac{3\pi R}{2}r + \frac{9\pi^3 R^2}{64}.$$

The vertex of this parabola is at $r = 3R/4$, so the minimum melting rate occurs at $r = 3R/4$ and the maximum at $r = 0$. The desired ratio is then $\dfrac{9\pi^3R^2/64}{9\pi^3R^2/64 - 9\pi R^2/16} = \boxed{\dfrac{\pi^2}{\pi^2 - 4}} \approx 1.681$

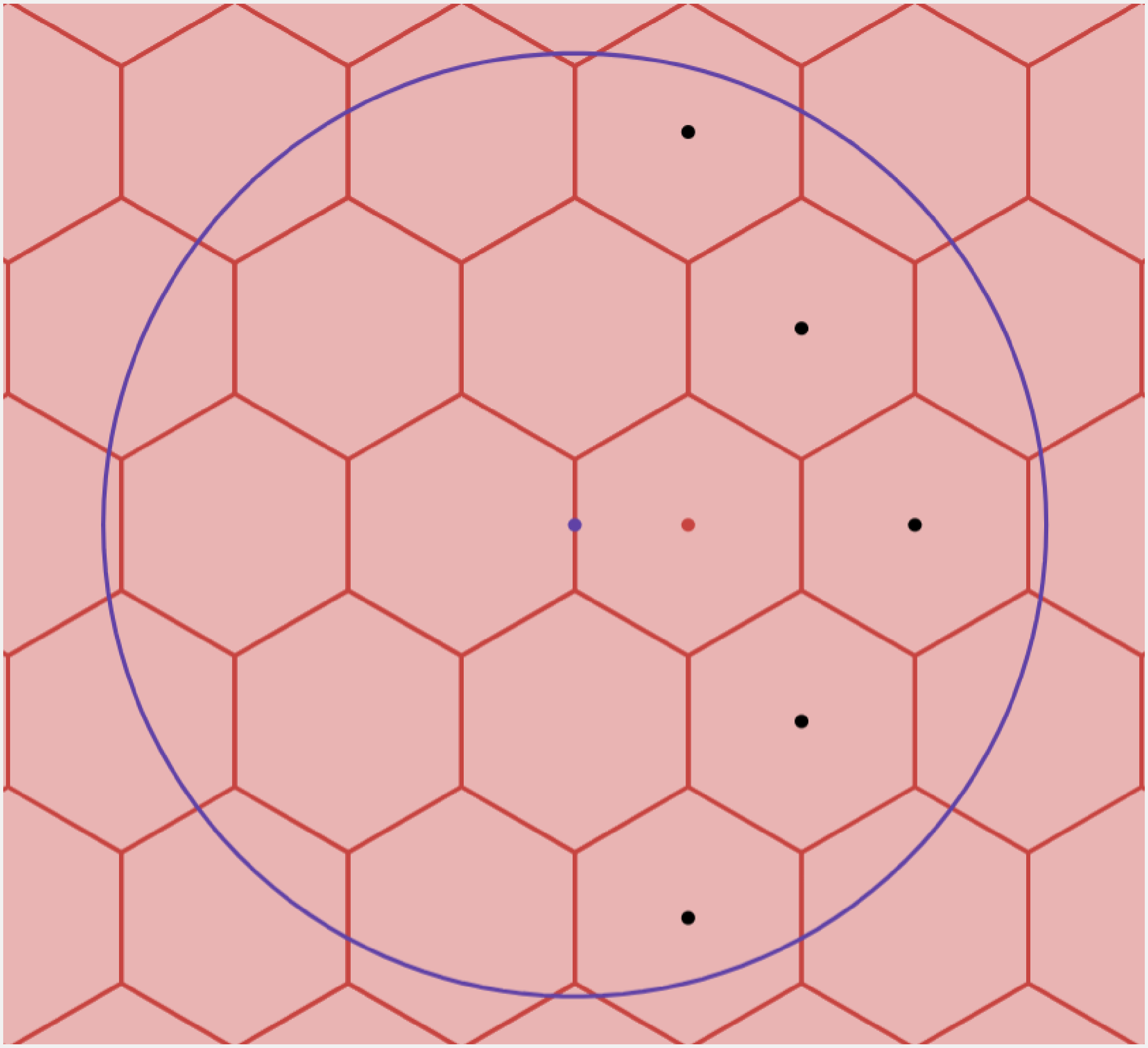
20. Hexagons are the Bestagons

Consider a two dimensional universe. In the centre of a hexagonal chamber with side length 1 m lies a 10 W light source. The interior of the chamber is perfectly reflective. What is the pressure at the centre of one of the walls 12 ns after the light is turned on? Note that pressure in this case indicates force per length.

Solution 20:

We consider the use of image sources. The setup is equivalent to asking what the combined pressure on the wall at the purple dot is from each of the shown sources, assuming the light can now pass through all the other walls. The red dot indicates the original hexagon.

The radius of the drawn circle is 3.6m, as the light has traveled that far in the given time.



The pressure from any source is given by

$$\frac{2I}{c}\cos^2\alpha = \frac{P}{\pi c}\frac{r_x^2}{r^3},$$

where r is the distance between the source and the center of the red wall. Taking the origin to be at the large blue dot, we can find r_x and r for each source. Note that the height of a hexagon is $\sqrt{3}$ times its side length.

Thus, we have

$$\frac{2P}{c}\sum\frac{r_x^2}{r^3} = \boxed{2.215\times10^{-8}\,\mathrm{N/m}}.$$

21. Spring-loaded capacitor

A capacitor is made of two flat metal plates, each of area $A = 20\ \text{mm}^2$, separated by an insulating spring of spring constant $k = 150$ N/m and natural length $d_0 = 0.01$ mm with no dielectric material contained within. The plates of the capacitor is then connected to a function generator which outputs a voltage $V(t)$, which consists of a DC component V_0. It is found that for some values of V_0 and some initial distances between the plate, the plate settles into an oscillation about a certain point. V_0 is increased until just below the point where this behaviour no longer applies, and then adjusted to half of this maximum value. The capacitor is then pulled apart to a distance y and released; the two plates slam together. Find the minimum y.

Solution 21: We first set up the Lagrangian for the plates. Suppose the distance between the plates is x; the capacitance is $\frac{A\epsilon_0}{x}$. Each plate has a charge of $\frac{A\epsilon_0 V}{x}$; considering the force between the plates gives an attractive force of $\frac{\epsilon_0 AV^2}{2x^2}$.

The equation of motion is therefore

$$2m\ddot{x} = -k(x - d_0) - \frac{A\epsilon_0 V^2}{2x^2} \tag{13}$$

We first wish to find the equilibria of the system. The equilibria satisfy the equation

$$x - d_0 = \frac{A\epsilon_0 V^2}{2kx^2} \tag{14}$$

The question gives that there is an equilibrium which is stable. Writing the system as a first-order equation:

$$2m\frac{d}{dt}\begin{pmatrix} x \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -k(x-d_0) - \frac{A\epsilon_0 V^2}{2x^2} \end{pmatrix} \tag{15}$$

The Jacobian of the RHS is

$$\begin{pmatrix} 0 & 1 \\ -k + \frac{A\epsilon_0 V^2}{x^3} & 0 \end{pmatrix} \tag{16}$$

The eigenvalues of the Jacobian satisfy

$$\lambda^2 = -k + \frac{A\epsilon_0 V^2}{x^2} = -k(3 - \frac{2d_0}{x}) \tag{17}$$

For stability to occur, we require that both eigenvalues are imaginary (so that the locality is neutrally stable); otherwise it must be unstable. We therefore demand $x > \frac{2d_0}{3}$. Checking the equation and substituting $x' = \frac{x}{d_0}$, the equilibrium condition is

$$x'^2 - x'^3 = \frac{A\epsilon_0 V^2}{2kd_0^3} \tag{18}$$

For there to exist a solution with $x' > \frac{2}{3}$, we require the RHS must be less than $\frac{4}{27}$. Under this situation the position will oscillate around the neutrally stable equilibrium point. The maximum voltage is therefore

$$V_{max} = (\frac{8kd_0^3}{27A\epsilon_0})^{\frac{1}{2}} = 15.84\ \text{V} \tag{19}$$

Half of this gives 7.92 V and hence an equilibrium condition of

$$x'^2 - x'^3 = \frac{1}{27} \tag{20}$$

The stable equilibrium is at $x' = 0.960$. The plate is then pulled apart and it subsequently slams together; this implies that it has sufficient energy (and will gain enough from the voltage source) to exceed the unstable equilibrium point at $x' = 0.218$.

The conserved quantity is

$$E = m\dot{x}^2 + \frac{1}{2}k(x - d_0)^2 - \frac{A\epsilon_0 V^2}{2x} \tag{21}$$

At $x = 0.218d_0$ and $\dot{x} = 0$, $E = 2.039 \times 10^{-9}$. Numerically solving the other solutions for x at this E and at $\dot{x} = 0$, we get $x' = 1.56$ as the other (non-repeat) solution. Hence $y = \boxed{1.56 \times 10^{-5}\ \text{m}}$.

22. Supernucleus

Scientists at CERN have recently been working on a novel super-nucleus - a nucleus with a mass number of $A = 16000$ and $Z = 1$ proton. Its radius is unusually small, at $R = 10^{-15}$ m. As is the natural decision to make, they've decided to place the super-nucleus in a box full of electron plasma at a temperature $kT = 100$ eV. Given the electron density of the plasma $n = 10^{19}\ \text{m}^{-3}$, what is the mean time between electron collision with the nucleus? Treat all particles classically.

Solution 22:
This is a scattering problem. Since we are working in a central field, we may conserve angular momentum around the center of mass between an electron and the nucleus. Since the nucleus is so heavy, we take the center of mass as simply at the nucleus. Let us first just consider the motion of one electron. ρ and v_∞ are defined as shown in the diagram. The angular momentum is $L = m_e \rho v_\infty$ and the energy is simply $E = m_e v_\infty^2/2$. While the motion progresses, the electron energy can be written as, dropping the subscript e for the mass:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r},$$

where r is the radial distance from the center of the nucleus. At the point of closest approach, $\dot{r} = 0$, and hence we have

$$E = \frac{L^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}.$$

Solving for L at $r = R$, we have

$$2mR^2E + \frac{e^2 mR}{2\pi\varepsilon_0} = L^2,$$

or,

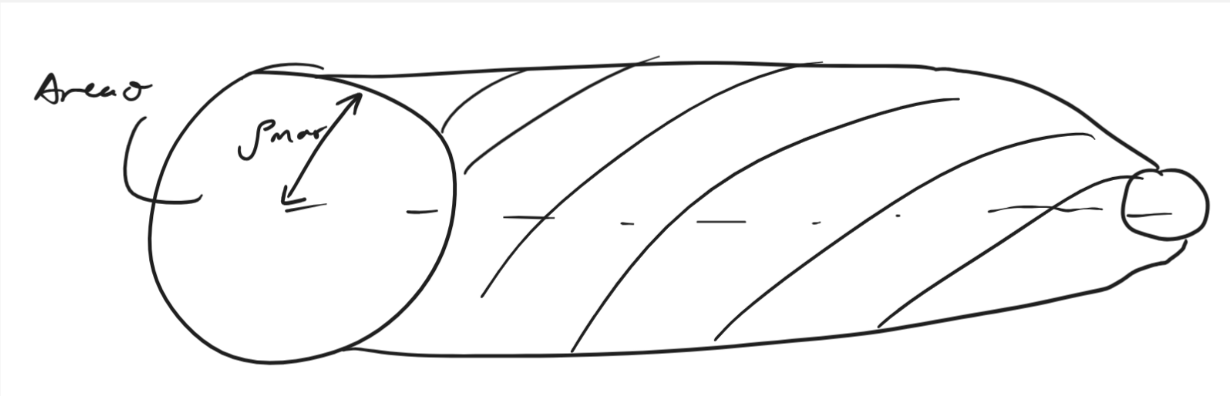
$$L = \sqrt{2mR^2E + \frac{e^2 mR}{2\pi\varepsilon_0}}.$$

Substituting our values for L and E,

$$mv_\infty\rho = \sqrt{m^2R^2v_\infty^2 + \frac{e^2 mR}{2\pi\varepsilon_0}}.$$

Solving for ρ now yields

$$\rho_{max} = R\sqrt{1 + \frac{e^2}{2R\pi\varepsilon_0 m v_\infty^2}}.$$



In this diagram, the central axis is parallel to the electron velocity. Any electron with velocity vector parallel to the cylinder axis that lies in the cylinder will strike the nucleus - hence, we can integrate over all velocities to find the time τ before the expected number of electrons in these volumes sums to 1. We must integrate over all speeds since the cross-sectional area σ varies with speed. Mathematically, with $f(v)$ as the Maxwellian speed distribution $f(v) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$, this can be expressed as the condition

$$n\int_0^\infty (v\tau)\sigma(v)f(v)\,\mathrm{d}v = 1$$

Substitution yields

$$n\tau\pi\int_0^\infty v\rho_{max}^2 \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}} v^2 \exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v = 1$$

$$4n\tau R^2\sqrt{\pi}\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\left[\int_0^\infty v^3\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v + \frac{e^2}{2\pi\varepsilon_0 Rm}\int_0^\infty v\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v\right] = 1$$

With the substitution $u = \frac{mv^2}{2kT}$, these integrals become

$$4n\tau\sqrt{\pi}R^2\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\left[\int_0^\infty 2\left(\frac{kT}{m}\right)^2 ue^{-u}\,\mathrm{d}u + \frac{e^2 kT}{2\pi\varepsilon_0 Rm^2}\int_0^\infty e^{-u}\,\mathrm{d}u\right] = 1$$

$$n\tau R^2\sqrt{\frac{2\pi kT}{m}}\left[2 + \frac{e^2}{2\pi\varepsilon_0 RkT}\right] = 1$$

Solving for τ yields

$$\tau = \boxed{(2nR^2\sqrt{\frac{2\pi kT}{m}}(1 + \frac{e^2}{4\pi\varepsilon_0 RkT}))^{-1}} = 0.330\text{s}$$

Note: We may neglect the shielding of the electric field due to the electron plasma as the Debye length (characteristic length for plasma shielding) is $\lambda_D = \sqrt{\frac{\varepsilon_0 kT}{ne^2}} \sim 10^{-5}\text{m} \gg R$

23. I-Conic

A solid cone of uniform density, with slant height 2 m and base radius 1 m, lies on its curved face in stable equilibrium on a slanted plane such that its axis is horizontal. There is a very high coefficient of friction between the cone and the plane. Find the frequency of small oscillations about this equilibrium.

Solution 23:
This problem is approached using energy methods. Note the height of the cone is $h = \sqrt{3}$ m and that the centre of mass of the cone is 75% the way from the tip. The half cone angle is $\alpha = 30°$.

The kinetic energy term can be expressed as

$$K = \frac{1}{2} I_r \omega_r^2,$$

where we consider rotation about the axis instantaneously at rest, and the contact line with the slope. In order to calculate the moment of inertia about the rotation axis, we must consider the moment of inertia tensor. For a cone, this can be expressed as

$$\mathbf{I} = \begin{bmatrix} \frac{3}{80}m(4r^2+h^2) & 0 & 0 \\ 0 & \frac{3}{80}m(4r^2+h^2) & 0 \\ 0 & 0 & \frac{3}{10}mr^2 \end{bmatrix},$$

where the z-axis lines up with the cone's axis. The unit direction about which the cone is rotating is given by

$$\mathbf{n} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix},$$

so it follows that

$$I_n = \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n} = \frac{3}{320}m(4r^2+h^2) + \frac{9}{40}mr^2 = \frac{21}{80}mr^2 + \frac{3}{320}mh^2,$$

through the center of mass of the cone. Accounting for the offset rotation axis, the parallel axis theorem states we must add on a term of md^2 giving

$$\frac{I_r}{m} = \frac{21}{80}r^2 + \frac{3}{320}h^2 + \left(\frac{3}{4}h\sin\alpha\right)^2 = \frac{57}{80}\ \mathrm{m}^2.$$

Note that $\omega_r = \dot{\theta}/\tan(\alpha) = \sqrt{3}\dot{\theta}$, where θ is the angle from equilibrium as measured in the plane. Thus, with the appropriate units, we have

$$K = \frac{1}{2}\frac{57}{20}m\dot{\theta}^2.$$

For the potential term, we observe that

$$U = mgL_{cm}(1-\cos\theta)\sin(\alpha) \approx mgL_{cm}\sin(\alpha)\frac{\theta^2}{2}.$$

As L_{cm} is taken in the plane,

$$L_{cm} = \frac{3}{4}h\cos\alpha = \frac{9}{8}\ \mathrm{m}.$$

Finally, we have

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgL_{cm}}{I_r}} = \boxed{0.626\ \mathrm{Hz}}.$$

24. This problem has been removed from the test.

Solution 24: This problem has been removed from the test.

25. Relativistic Race

A relativistic tortoise and rabbit are racing. Both animals start running from rest at $t = 0$ in the lab frame along a one-dimensional racetrack. To see who wins the race, Biologist Billy gives each animal a stopwatch initially set to 0, which they start at $t = 0$ and stop when they cross the finish line. Unfortunately, when the animals return the stopwatches to Billy, he finds that they both read the same value! The tortoise's proper acceleration is $a_t = 1.10 \cdot 10^6$ m/s^2, and the rabbit's proper acceleration is $\frac{c}{t_r+k_2}$, where t_r is the rabbit's proper stopwatch reading and $k_2 = 190$ s. In the lab frame, who won the race, and by how much time?

Input the positive value in seconds for the margin of victory if the tortoise won and the negative value if the rabbit won. You may find an online integral calculator such as Wolfram Alpha or this one to be helpful.

Solution 25: First, we will find the velocity as a function of time elapsed in the animals' frame. By working in the instantaneously comoving frame, we obtain this general result:

$$v(t+\mathrm{d}t) = \frac{v(t)+a\mathrm{d}t}{1+v(t)a\mathrm{d}t/c^2} \approx (v(t)+a\mathrm{d}t)(1-\frac{v(t)a\mathrm{d}t}{c^2}) \approx v(t) - \frac{(v(t))^2 a\mathrm{d}t}{c^2} + a\mathrm{d}t$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a(1-\frac{v^2}{c^2})$$

$$v(t) = c\tanh\left(\frac{1}{c}\int_0^t a(t)\mathrm{d}t\right).$$

Let t_t represent the time elapsed in the tortoise's frame. Then,

$$v_t = c\tanh\left(\frac{a_t t_t}{c}\right).$$

Likewise,

$$v_r = c\tanh\left(\log\left(1+\frac{t_r}{k_2}\right)\right).$$

We must solve for the time t_f it takes the animals to cross the finish line in their rest frames. To do this, we use the condition that the race's distance is constant in the lab frame.

$$\int_0^{t_f} \frac{v_r}{\sqrt{1-v_r^2/c^2}} dt_r = \int_0^{t_f} \frac{v_t}{\sqrt{1-v_t^2/c^2}} dt_t$$

$$\int_0^{t_f} \sinh\left(\log\left(1+\frac{t_r}{k_2}\right)\right) dt_r = \int_0^{t_f} \sinh\left(\frac{a_t t_t}{c}\right) dt_t$$

$$\frac{t_f}{2} + \frac{t_f^2}{4k_2} - \frac{k_2}{2}\ln\left(1+\frac{t_f}{k_2}\right) = \frac{c}{a_t}(\cosh\left(\frac{a_t t_f}{c}\right) - 1)$$

Solving the above equation numerically, we find $t_f = 281.65536$ s.
Now we must find the relationship between the times elapsed in the animals' frames with the time elapsed in the lab frame. Let the primed values represent the time elapsed in the lab frame when the animals cross the finish line. For the tortoise:

$$\begin{aligned} \mathrm{d}t &= \gamma_t \mathrm{d}t_t \\ \int_0^{t'_t} \mathrm{d}t &= \int_0^{t_f} \frac{1}{\sqrt{1-v_t^2/c^2}} \mathrm{d}t_t \\ t'_t &= \int_0^{t_f} \cosh(a_t t_t/c)\mathrm{d}t_t \\ &= \frac{c}{a_t}\sinh\left(a_t t_f/c\right). \end{aligned}$$

Similarly, for the rabbit:

$$\begin{aligned} \int_0^{t'_r} \mathrm{d}t &= \int_0^{t_f} \frac{1}{\sqrt{1-v_t^2/c^2}} \mathrm{d}t_r \\ t'_r &= \int_0^{t_f} \cosh\left(\log\left(1+\frac{t_r}{k_2}\right)\right)\mathrm{d}t_r \\ &= \frac{t_f}{2} + \frac{t_f^2}{4k_2} + \frac{k_2}{2}\log(1+t_f/k_2). \end{aligned}$$

Finally,

$$t'_r - t'_t = \frac{t_f}{2} + \frac{t_f^2}{4k_2} + \frac{k_2}{2}\log(1+t_f/k_2) - \frac{c}{a_t}\sinh(a_t t_f/c) = \boxed{-2.875\,\mathrm{s}}.$$

The answer is negative, so the rabbit wins the race.

26. Glass Block

Steve holds a long square prism with refractive index 3/2 in midair such that its axis is horizontal. Sunlight is shining on the block. If the magnitude of the radiative force exerted on the block is at half of its maximum possible value, find the smallest angle of a block face to the plane formed by the line from the Sun to the centre of the prism and the horizontal (i.e. the axis of the prism). Assume that no light is reflected when entering the block.

Solution 26: We first need to find the trajectory of light rays passing through the block. We consider the 2D simplification since there are two faces the light ray is always parallel to. The remaining four faces are described by the line segments $y = \pm 0.5$ from $x = -0.5$ to $y = 0.5$, and vice versa. The incoming light ray is described by an arbitrary straight line $y = mx + c$. We assume without loss of generality that the light ray impacts the right side face, i.e. $-0.5 < 0.5m + c < 0.5$. We further assume without loss of generality that $m > 0$.

The angle of the light ray to the normal is $\theta_0 = \tan^{-1} m$. The angle of the outgoing ray once refracted is to the normal is hence $\theta_1 = \sin^{-1}(2\sin\theta_0/3)$. The slope of the resulting line is hence

$$m' = \tan\theta_1 = \frac{2\sin\theta_0/3}{\sqrt{1-4\sin^2\theta_0/9}} = \frac{1}{\sqrt{9(1+\cot^2\theta_0)/4-1}} = \frac{2m}{\sqrt{5m^2+9}}. \tag{22}$$

The equation describing the light ray inside the cube is

$$y = \frac{2m}{\sqrt{5m^2+9}}(x-0.5) + 0.5m + c. \tag{23}$$

We now have two cases. In the first case, the light ray hits the left side face, in which case we have the inequality

$$-\frac{2m}{\sqrt{5m^2+9}} + 0.5m + c > -0.5. \tag{24}$$

Light rays satisfying this have outgoing rays parallel to ingoing rays and do not contribute to the radiative force.

On the other land, light rays may also hit the bottom face. In this case rays must satisfy

$$-1 < \frac{(-0.5-0.5m-c)\sqrt{5m^2+9}}{2m} < 0, \tag{25}$$

and they intersect the face at the point $(-(0.5+0.5m+c)\sqrt{5m^2+9}/2m, -0.5)$. We note that the critical angle is $\cot\theta_c = \sqrt{5}/2$ and $2m/\sqrt{5m^2+9} < 2/\sqrt{5} < \sqrt{5}/2$, so total internal reflection always occurs. The ray then undergoes a symmetric process where it hits the left face and is refracted out into air with slope $-m$.

In this case, the fractional momentum of the light ray which is converted to radiative force is equal to twice its y-component:

$$f(m) = 2\sin\theta_0 = \frac{2m}{\sqrt{1+m^2}}. \tag{26}$$

To find the total force we integrate over the permissible range of c where the light ray contributes to the force:

$$-0.5-0.5m < c < \frac{2m}{\sqrt{5m^2+9}} - 0.5m - 0.5. \tag{27}$$

The force in the x-direction for light rays hitting the cube at an angle characterised by m is hence

$$F_x = \frac{I}{c}lf(m)\frac{2m}{\sqrt{5m^2+9}} = \frac{Il}{c}\frac{2m^2}{\sqrt{(5m^2+9)(1+m^2)}}. \tag{28}$$

This considers the forces exerted in one direction by the light rays. However, the light rays will also be hitting the upper face. We can, by symmetry, replace m by $1/m$ in the above analysis and arrive at the force exerted in the y-direction. Note that I in each case refers to the total intensity of the light hitting the face, which is proportional to the projected area of the face in the direction of the light beam - another factor of $\cos\theta_0$. The magnitude of force exerted in total is

$$F^2 = \left(\frac{Il}{c}\right)^2\left(\frac{4m^4}{(5m^2+9)(1+m^2)^2} + \frac{4m^{-4}}{(5m^{-2}+9)(1+m^{-2})^2}\right) \tag{29}$$

By using symmetry, the fraction is symmetric about $m = 1$. Numerical testing quickly shows this is a maximum point, with the value of $1/7$.

The maximum force exerted is therefore

$$F = \frac{1}{\sqrt{7}}\frac{I}{c}. \tag{30}$$

We want to find the m at which the function is equal to $1/28$. Numerically solving, we get $m = 0.229$ (or equivalently $m = 4.367$) which gives an angle of $\boxed{12.9^\circ}$ (or equivalently $\boxed{77.1^\circ}$).

27. Sproing

Two point masses of mass $M = 59$ kg are attached to opposite ends of a spring with nonzero rest length. The spring has mass $m = 2$ kg and spring constant $k = 9$ N/m. The system is placed in a frictionless and massless tube which rotates around the center of mass of the system with a **fixed** angular velocity. In the ensuing oscillation, suppose the orbital path of each of the masses is periodic without intersecting with itself. Find the maximum angular velocity for which this is possible.

Solution 27:
Since the motion is periodic, standing waves appear. Also, due to the constraining tube, the Coriolis force can be ignored. Suppose the spring has rest length L. Let's first find the spring shape in equilibrium; we will define a function $y_0(x)$ to denote the location of a point as a function of its original position x when the spring is relaxed, relative to the center of mass. The force on adjacent infinitesimally small springs is given by

$$\begin{aligned} F &= \frac{km}{\mathrm{d}m}(\mathrm{d}y_0 - \mathrm{d}x) \\ &= kL(y_0' - 1), \end{aligned}$$

since the original mass density $\lambda = \mathrm{d}m/\mathrm{d}x = m/L$ and $k_{\text{eff}} = km/\mathrm{d}m$. Working in the rotating frame and balancing forces on a small spring of mass $\mathrm{d}m$,

$$\begin{aligned} -\frac{m\omega^2 y_0 \mathrm{d}x}{L} &= \mathrm{d}F \\ -\frac{m\omega^2 y_0}{L} &= kLy_0''. \end{aligned}$$

We recognize that this differential equation has the same form as a harmonic oscillator, so

$$y_0 = A\sin\left(\frac{x\varphi}{L}\right),$$

where $\varphi = \sqrt{m\omega^2/k}$. Balancing forces on one of the masses,

$$\begin{aligned} M\omega^2 A\sin\left(\frac{\varphi}{2}\right) &= kL\left(A\varphi\cos\left(\frac{\varphi}{2} - 1\right)\right) \\ A &= \frac{L}{\varphi\cos(\varphi/2) - \varphi^2\sin(\varphi/2)} \\ y_0(x) &= \frac{L}{\varphi\cos(\varphi/2) - \varphi^2\sin(\varphi/2)}\sin\left(\frac{x\varphi}{L}\right). \end{aligned}$$

Now we will move onto an analysis of when the spring is in motion. Define a function $y(x, t)$ similar to $y_0(x)$, but now additionally as a function of time. Since we know that the spring oscillates, it will take a form of

$$y(x, t) = y_0 + B\sin(\omega_t t)f(x).$$

Writing $F_{net} = ma$ for an infinitesimal segment of the spring,

$$\begin{aligned}
\frac{m\mathrm{d}x}{L}\frac{\partial^2 y}{\partial t^2} &= \mathrm{d}F + \frac{m\omega^2 y \mathrm{d}x}{L} \\
\frac{\partial^2 y}{\partial t^2} &= \frac{kL^2}{m}\frac{\partial^2 y}{\partial x^2} + \omega^2 y \\
-B\omega_t^2 f \sin(\omega_t t) &= \frac{kL^2}{m}\left(-\frac{A\varphi^2}{L^2}\sin\frac{x\varphi}{L} + Bf''\sin(\omega_t t)\right) + \omega^2\left(A\sin\frac{x\varphi}{L} + Bf\sin(\omega_t t)\right) \\
f'' &= -f\frac{m(\omega^2+\omega_t^2)}{kL^2}.
\end{aligned}$$

This differential equation again has the same form as a harmonic oscillator, so

$$f = \sin\left(\frac{x}{L}\sqrt{\psi^2+\varphi^2}\right),$$

where $\psi = \sqrt{m\omega_t^2/k}$. Writing $F_{net} = ma$ for one of the point masses this time, recalling our original expression for F,

$$\begin{aligned}
M\frac{\partial^2 y}{\partial t^2}\bigg|_{x=L/2} &= -F + m\omega^2 y \\
M\frac{\partial^2 y}{\partial t^2}\bigg|_{x=L/2} &= -kL\left(\frac{\partial y}{\partial x}\bigg|_{x=L/2} - 1\right) + m\omega^2 y(L/2,t) \\
-MA\omega_t^2\sin(\omega_t t)\sin\left(\frac{\sqrt{\psi^2+\varphi^2}}{2}\right) &= -kL\left(\frac{A\varphi}{2}\cos\left(\frac{\varphi}{2}\right)\right. \\
&\quad \left. + B\sqrt{\psi^2+\varphi^2}\sin(\omega_t t)\cos\left(\frac{\sqrt{\psi^2+\varphi^2}}{2}\right) - 1\right) \\
&\quad + m\omega^2\left(A\sin\left(\frac{\varphi}{2}\right) + B\sin(\omega_t t)\sin\left(\frac{\sqrt{\psi^2+\varphi^2}}{2}\right)\right) \\
k\sqrt{\psi^2+\phi^2} &= (M\omega_t^2 + m\omega^2)\tan\left(\frac{\sqrt{\psi^2+\varphi^2}}{2}\right). \qquad (31)
\end{aligned}$$

You can check that this equation correctly reduces to the common non-rotating form when taking the limit as $\omega \to 0$. Moving on, notice that $f(x)$ can decrease as x increases, appearing on the spring quicker than $y_0(x)$ and occurring when $\sqrt{\psi^2+\varphi^2} = \pi$. This is obviously not physical[a], so we have the condition

$$\sqrt{\psi^2+\varphi^2} < \pi$$

for all values of ω. From polar graphs, for the orbits of the point masses to be periodic and not intersect with itself, we have the relation $\omega_t = n\omega$, where n is an integer; the maximum possible ω then occurs when $n = 1$ since the mass is stated to be oscillating. After substituting $\psi = \varphi$ into 31, we finally numerically find that $\omega = \boxed{0.54026 \text{ rad/s}}$[b].

[a] Mathematically, it's due to how both the centrifugal and spring force depend on y. For some initial conditions, the centrifugal force overpowers the spring force, so the expression attempts to compensate by making the spring appear to turn around.

[b] Remark: due to how the selected values of M and m (unintentionally) implied a possible approximation, many teams submitted values around 0.55 rad/s. Additionally, assuming no oscillations also yielded that result. This value is not within 1% accuracy, so thus this problem was one of the least solved.

28. Five Big Booms

A spherically symmetric explosion occurs at a point along the axis of a very long cylindrical chamber with radius R. A sensor is a distance R along the axis from the explosion and can be modeled as a sphere of radius $r \ll R$. Given that $r/R = 10^{-30}$, find the proportion of energy eventually absorbed by the sensor to within 1% relative accuracy. The walls of the chamber are perfectly reflecting.

Solution 28: First, note that the primary sound contributes a negligible amount of energy since the corresponding solid angle is proportional to $(r/R)^2$ while the solid angles of the echoes are proportional to r/R. Furthermore, the nth echo occurs at an angle $\tan(\theta_n) = 2n$ from the axis, so it takes up solid angle

$$\Omega_n \approx 2\pi \sin(\theta_n) \cdot \frac{2r\cos(\theta_n)}{R} = \frac{8\pi r n}{R(1+4n^2)}.$$

If we summed these up for all n, they would diverge because the solid angles of different echoes begin to overlap. Note that echoes i and $i+1$ overlap if the following condition holds:

$$\theta_{i+1} - \theta_i < \frac{1}{2}\left(\frac{2r\cos(\theta_i)}{R} + \frac{2r\cos(\theta_{i+1})}{R}\right).$$

Since $\theta_{i+1} - \theta_i \ll 1$, we can rewrite this as

$$\frac{d\theta}{di} < \frac{2r\cos(\theta)}{R} \implies \frac{2}{1+4i^2} < \frac{2r}{R\sqrt{1+4i^2}} \implies i > \frac{1}{2}\sqrt{\frac{R^2}{r^2} - 1} \approx \frac{R}{2r}.$$

Thus, the whole hemisphere past $\tan(\theta_c) = R/r$ eventually hits the observer, contributing a solid angle $2\pi\cos(\theta_c) \approx 2\pi r/R$. The contributions of the echoes with $\theta < \theta_c$ are:

$$\sum_{i=1}^{R/2r} \Omega_i = \sum_{i=1}^{R/2r} \frac{8\pi r i}{R(1+4i^2)} = \frac{2\pi r}{R}\sum_{i=1}^{R/2r} \frac{i}{i^2 + 1/4} \approx \frac{2\pi r}{R}\left(\ln\left(\frac{R}{2r}\right) + \alpha\right)$$

Here, we use the asymptotic:

$$\sum_{i=1}^{N} \frac{i}{i^2+1/4} = \ln(N) + \alpha + O(1/N), \quad \alpha = 0.329\ldots$$

Thus, the fraction of energy hitting the observer is approximately

$$\frac{1}{4\pi}\left(\frac{2\pi r}{R}\left(\ln\left(\frac{R}{2r}\right) + \alpha\right) + \frac{2\pi r}{R}\right) = \boxed{\frac{r}{2R}\left(\ln\left(\frac{R}{2r}\right) + \alpha + 1\right)} = 3.486 \times 10^{-29}.$$

Remark: Approximating the sum as harmonic does not change the answer to within 1% accuracy.

29. Hang in there!

Consider a very long cylindrical neodymium magnet with radius $R = 0.450$ m, uniform longitudinal magnetisation $M = 5.00 \times 10^5$ A/m, and density $\rho = 7500$ kg/m^3. The magnet is held vertically by its top end. To within 1% relative accuracy, find the distance L from the bottom of the magnet where there is no tensile stress in the material (averaged over the cross-section).

Solution 29:
It's easiest to approach this question using the magnetic pole formalism. Here, a uniform magnetization $\vec{M}$ creates a "magnetic surface charge" $\sigma_m = \vec{M} \cdot \hat{n}$ at a boundary. Using an analogy to a parallel-plate capacitor, the force on the upper end of the segment of length L is

$$F_0 = \frac{1}{2}\mu_0 \pi R^2 M^2.$$

The force on the lower end of the segment can be approximated as point charges with magnitude $q_m = \pi R^2 M$, interacting with a force of:

$$\begin{aligned} F_1 &= -\frac{\mu_0}{4\pi}\frac{q_m^2}{L^2} \\ &= -\frac{1}{4}\mu_0 \pi R^2 M^2 \left(\left(\frac{R}{L}\right)^2 + O\left(\frac{R^4}{L^4}\right)\right) \end{aligned}$$

We have included a big-O term to represent the order of accuracy of our answer. For a first estimate, we ignore F_1 as F_1 is on the order of $F_0 R^2/L^2$. The assumption is valid if R^2/L^2 is very small.

Equating F_0 to gravitational force,

$$\begin{aligned} F_0 &= \frac{1}{2}\mu_0 \pi R^2 M^2 \\ &= \rho \pi R^2 L g \\ \therefore L &= \frac{1}{2}\mu_0 \frac{M^2}{\rho g} \\ &\approx 2.1371\,\mathrm{m} \end{aligned}$$

It seems that R^2/L^2 is on the order of 4%, which is not negligible. Thus, including the 2nd order term, the total force is

$$\begin{aligned} F &= \frac{1}{2}\mu_0 \pi R^2 M^2 \left(1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 + O\left(\frac{R^4}{L^4}\right)\right) \\ &= \rho g \pi R^2 L \\ \therefore L &= \frac{1}{2}\mu_0 \frac{M^2}{\rho g}\left(1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 + O\left(\frac{R^4}{L^4}\right)\right) \\ &\approx \boxed{2.0875\,\mathrm{m}}. \end{aligned}$$

We solve the last equation numerically. This is likely to be within 1% of the actual answer, and it turns out that it is.
Note: The last equation has two solutions! However, the other solution does not fit the $R^4/L^4 \ll 1$ approximation, which means that it is unphysical. To confirm that, you can look at the exact solution (explained below), and realise that the other solution doesn't exist at all.

(Optional) Obtaining the exact answer

This is not necessary, but it is also possible to obtain the exact length. This paper gives an exact expression for the force between two coaxial charged disks. Borrowing the result, we plot it on desmos and find that the length required is $\boxed{L = 2.09078\text{ m}}$. This is about 0.16% away from the 2nd order answer.

30. Fly Away

In the lab frame, two birds fly counterclockwise with speed u around a circle with radius R, starting an angle $\pi/2$ apart. An observer moves at speed v relative to the lab frame. In the observer's frame, the maximum distance between the birds is s. Over all values $u, v < c$, find the maximum of s/R.

Solution 30: Take $R = 1$, $c = 1$. Note that s^2 is the invariant interval between the leading bird and the trailing bird in the observer's frame. If these two events are separated by time Δt in the lab frame, the angle between them is $\pi/2 + u\Delta t$, so the distance between them is $d = 2\sin\left(\frac{\pi}{4} + \frac{u\Delta t}{2}\right)$. Then, we have:

$$s^2 = d^2 - \Delta t^2 = 4\sin^2\left(\frac{\pi}{4} + \frac{u\Delta t}{2}\right) - \Delta t^2$$

Now, we have two cases. If $\Delta t > \pi/2$, we can choose u such that the sine is 1. Then, $s^2 = 4 - \Delta t^2 < 4 - (\pi/2)^2$. Because $v = 0$ gives $s^2 = 2$, this is not optimal.

Thus, $\Delta t < \pi/2$, so we should take $u \to 1$. Then, $s^2 = 4\sin^2\left(\frac{\pi}{4} + \frac{\Delta t}{2}\right) - \Delta t^2$, which has a maximum of $s^2 \approx 2.801$ at $\Delta t \approx 0.7391$. This gives $\boxed{s \approx 1.674}$.

We claim that this maximum is attainable. Consider a moment in the observer's frame where the birds have the same y-coordinate and the leading bird's x-coordinate is larger (so $\Delta t > 0$). This must happen if we move from the leading bird having smaller to larger y-coordinate. If we take $v \to 1$, we find $s^2 \to 0$ at this moment by length contraction. Here, Δt takes its maximum possible value $\Delta t_+ \approx 1.962$. Similarly, the minimum possible value $\Delta t_- \approx -0.7749$ is achieved. Thus, all values $\Delta t_- < \Delta t < \Delta t_+$ are achieved by continuity.

31. Unconventional Oscillations

A thin uniform wire with mass per unit length $\lambda = 5.0 \times 10^{-5}$ kg/m is shaped into a circular loop and carries a constant current $I = 10$ A. It is placed in a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$, with its axis of rotational symmetry parallel to $\hat{\mathbf{x}}$. The wire loop is given an initial angular velocity $\boldsymbol{\omega_0} = \omega_0\hat{\mathbf{z}}$ about its center. Curiously, when $|\omega_0| < \omega_c$, the loop never makes a full rotation with respect to its center. Find ω_c in rad/s.

The magnitude of the electric field is $E = 1000$ V/m. Assume that the experiment is conducted in a vacuum, and there is no external gravitational or magnetic field.

Solution 31: Using the duality between electric and magnetic fields, the situation is identical to that of an electric dipole in a magnetic field.

We can use a modified result from APhO 2001 T2 (or EuPhO 2022 T3). For an electric dipole with dipole moment p, mass m and moment of inertia $I_\perp$ about its perpendicular axis, the critical angular velocity in a magnetic field with magnitude B is

$$\omega_c = \frac{2pB}{\sqrt{I_\perp M}} = 2\sqrt{2}\frac{pB}{MR} \quad \left(\because I_\perp = \frac{1}{2}I_\parallel = \frac{1}{2}MR^2\right).$$

Using $p = \pi R^2 I$,

$$\omega_c = 2\pi\sqrt{2}\frac{RIB}{M}.$$

This is off by some power of μ_0 and ϵ_0. By dimensional analysis, we have to multiply the above by a factor of $\mu_0\epsilon_0 = c^{-2}$.

$$\omega_c = \sqrt{2}\frac{2\pi RIE}{Mc^2} = \boxed{\sqrt{2}\frac{IE}{\lambda c^2}} = 3.14\times 10^{-9}\,\text{rad/s}.$$

This is a very small number (equivalent period is about 63 years), as it is fundamentally due to the relativistic hidden momentum, which results in a very small charge buildup on one side of the magnet. Note that we can still use Newtonian dynamics as the speed of the magnet is very small compared to the speed of light. **Method 2: Hidden momentum**

The hidden momentum in a magnetic dipole is

$$\mathbf{p}_{\text{hid}} = \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}).$$

By conservation of momentum,

$$\mathbf{p}_{\text{tot}} = M\mathbf{v} + \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}) = \text{constant}.$$

One might wonder why the EM field momentum is ignored here. The reason is that the EM field momentum is actually transferred to whatever is creating the E field! As the wire rotates, the changing B field creates an induced E field that exerts a force on whatever is creating the external E field. Mechanical energy is also conserved, as there is no energy stored in the electric field interaction. We can prove this by finding the power of the electric field on a unit charge: $P = \mathbf{E}\cdot\mathbf{v}$. Since the CM does not move parallel to the electric field, the summation of the power over all charges must be 0. Hence,

$$\frac{1}{2}Mv^2 + \frac{1}{2}I_\perp\omega^2 = \text{constant}.$$

The maximum speed is when the magnetic dipole points in the opposite direction to the initial direction. Then,

$$Mv_{\max} = \frac{2}{c^2}mE.$$

In the limiting case, the angular velocity here is 0. If the angular velocity were any lower, conservation of energy would prevent the dipole from ever reaching this position.

$$\frac{1}{2}Mv_{\max}^2 = \frac{1}{2}I_\perp\omega_c^2.$$

Using $I_\perp = MR^2/2$,

$$\omega_c = \frac{2\sqrt{2}}{c^2}\frac{mE}{MR}.$$

Then using $m = \pi R^2 I$ and $M = 2\pi R\lambda$,

$$\omega_c = \sqrt{2}\frac{IE}{\lambda c^2} = 3.14 \times 10^{-9} \text{ rad/s}.$$

32. Star Power

Fusion takes place when two protons are less than 1.5 fm apart. One gram of hydrogen gas at room temperature and pressure is adiabatically compressed by lasers in a fusion reactor into a volume of 10^{-10} m^3. Find the initial output power of the fusion reactor. You may assume that other than the proton-deuteron fusion reaction, $H + H \to D + \nu$, no further fusion reactions take place. **The accuracy of this problem should be within 5%**.

Solution 32:
We first need to find the temperature and number density of the gas of hydrogen *atoms* (as the rapid compression will dissociate all chemical bonds). We may assume that the final temperature is much higher than the phase transition temperature so the majority of the adiabatic heating takes place during the monoatomic phase. With this, we have initially a volume of gas of $V_0 = \frac{0.5 \text{ mol} \times R \times 300 \text{ K}}{10^5 \text{ Pa}} = 0.0125$ m^3, and the monoatomic adiabatic index is $\frac{5}{3}$. Hence the final temperature is

$$T_f = T(\frac{V}{V_0})^{1-\gamma} = 7.49 \times 10^7 \text{ K} \tag{32}$$

and the final number density is

$$n_f = \frac{N_A}{V} = 6.02 \times 10^{33} \text{ m}^{-3} \tag{33}$$

(This is, of course, unrealistically high for a fusion reactor.) If we model the protons as hard spheres which are 1.5 fm in diameter, then we get a rate of collision of $\frac{1}{\sqrt{2}} N n \pi r^2 v$. However, even when two protons are on a direct collision trajectory, they need to overcome the Coulomb barrier in order to actually collide. We need to estimate the probability of this happening.

From the WKB approximation (and also a generalisation of the plane wavefunction), the probability of a particle tunneling through a potential is given by

$$Pr = e^{-2\int_{r_0}^{r_1} \sqrt{\frac{2\mu}{\hbar^2}(V(r)-E)}dr} \tag{34}$$

where r_0 and r_1 are the inner and outer equivalence points - the points at which $V(r) = E$, the kinetic energy of the particle. As with any two-body system the two body case is reduced to a particle scattering off a potential barrier through the use of the reduced mass $\mu = \frac{m_p}{2}$. Moreover the average kinetic energy of the protons is $\frac{3}{2}kT = 2.46 \times 10^{-15}$ J and the magnitude of the Couloumb barrier is $\frac{1}{4\pi\epsilon_0}\frac{e^2}{1.5 \text{ fm}} = 1.54 \times 10^{-13}$ J, much larger than the kinetic energy. We hence neglect E except in determining r_1. Since $r_1 >> r_0$ also, we can neglect the latter. The exponent is now

$$\log Pr = -2\int_0^{\frac{e^2}{4\pi\epsilon_0 E}} \sqrt{\frac{m_p}{\hbar^2}\frac{e^2}{4\pi\epsilon_0 r}}dr = -4\sqrt{\frac{m_p e^2}{4\pi\hbar^2\epsilon_0}}\sqrt{\frac{e^2}{4\pi\epsilon_0 E}} = -\sqrt{\frac{2}{v^2}}\frac{e^2}{\pi\hbar\epsilon_0} \tag{35}$$

We wish therefore to evaluate the average over the Boltzmann distribution of the quantity $v\exp\left(-\frac{\sqrt{2}e^2}{\pi\hbar\epsilon_0 v}\right)$. After some numerical manipulation we find the quantity averages out to 1514.

Therefore, the rate of collisions leading to fusion is $\frac{1}{\sqrt{2}}Nn\pi r^2 \times 1514 = 1.219 \times 10^{31}$. Each fusion reaction releases an energy equivalent to the difference in masses of deuterium and the two protons, which is 6.73×10^{-14} J, plus a positron and electron which annihilate, releasing a further 1.64×10^{-13} J for a total of 2.31×10^{-13} J. This multiplied by the rate of fusion occurring gives a power of $\boxed{2.82 \times 10^{18}\ \text{W}}$. This is on par with the power output of a nuclear detonation. (Note: due to the nature of semiclassical approximations in the problem, a 5% tolerance should be given to the answer.)

33. Party Pooper

A balloon with initial radius $r_0 = 5.00\,\text{cm}$ is made of a rubber film with constant surface tension $\sigma = 10.0\,\text{N/m}$. Houlai inflates the balloon with a very small hand pump; the pump injects air into the balloon once it reaches the balloon's pressure. If the balloon's final radius is $r_1 = 20.0\,\text{cm}$, find the difference between the volume of air used by the pump and the volume change of the balloon. Assume that air is a diatomic ideal gas with pressure $p_0 = 1.00 \cdot 10^5$ Pa and that there is no heat transfer across the balloon or pump surfaces.

Solution 33: Let the balloon have radius r and pressure P. Let the pump's capacity be dV. Let's consider how the pump works. First, it compresses adiabatically from P_0 to P, so its final volume is $dV_f = \left(\frac{P_0}{P}\right)^{\frac{1}{\gamma}} dV$. Then, this gas is added to the balloon at constant pressure P. Let the internal energy of the balloon's gas be U. Then, we have:

$$dU = \frac{1}{\gamma - 1}P\ dV_f + P(dV_f - dV_b) = \frac{\gamma}{\gamma-1}P\ dV_f - P\ dV_b$$

The first term is the internal energy of the gas in the pump and the second term is the work done on the gas during the injection into the balloon, where V_b is balloon's volume. Thus:

$$dU = \frac{\gamma}{\gamma-1}P_0\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} dV - 4\pi r^2 P\ dr$$

Furthermore, $U = \frac{1}{\gamma-1}PV_b = \frac{4\pi}{3(\gamma-1)}Pr^3$, so $dU = \frac{4\pi}{3(\gamma-1)}(r^3\ dP + 3r^2 P\ dr)$, giving:

$$4\pi r^2\left(\frac{1}{3(\gamma-1)}r\ dP + \frac{\gamma}{\gamma-1}P\ dr\right) = \frac{\gamma}{\gamma-1}P_0\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} dV$$

By the Young-Laplace equation, we have $P = P_0 + \frac{2\sigma}{r}$, so $dP = -\frac{2\sigma}{r^2}\ dr$. Thus:

$$4\pi r^2\ dr\left(-\frac{2}{3(\gamma-1)}\frac{\sigma}{r} + \frac{\gamma}{\gamma-1}\left(P_0 + \frac{2\sigma}{r}\right)\right) = \frac{\gamma}{\gamma-1}P_0\left(1 + \frac{2\sigma}{P_0 r}\right)^{\frac{\gamma-1}{\gamma}} dV$$

$$\implies dV = 4\pi r^2\ dr\left(1 + \left(2 - \frac{2}{3\gamma}\right)\frac{\sigma}{P_0 r}\right)\left(1 + \frac{2\sigma}{P_0 r}\right)^{-\frac{\gamma-1}{\gamma}} \approx 4\pi r^2\ dr\left(1 + \frac{4}{3\gamma}\frac{\sigma}{P_0 r}\right)$$

Here, we use the fact that $\frac{\sigma}{P_0 r} \ll 1$ to the desired accuracy. Finally:

$$\Delta V - \Delta V_b = \int\left(dV - 4\pi r^2\ dr\right) \approx \int_{r_0}^{r_1}\frac{16\pi\sigma}{3\gamma P_0}r\ dr = \boxed{\frac{8\pi\sigma}{3\gamma P_0}(r_1^2 - r_0^2)} = 2.24 \cdot 10^{-5}\,\text{m}^3$$

34. A Cloudy Day

After an asteroid impact the middle of a large ocean, a hypercane is forming. The surface temperature is measured to be 320 K and the pressure to be 10^5 Pa. By considering what happens to a parcel of air picking up moisture from the ocean surface as it rises adiabatically in the atmosphere, find to within 10% accuracy the highest altitude at which clouds can form. The latent heat of water is 2260 kJ/kg. You may assume that water vapour is an ideal gas with molar mass 18 g/mol and dry air is a diatomic ideal gas with molar mass 29 g/mol. Also assume that relative humidity may not exceed 100%.

Solution 34: We first need to consider the change in the vapour pressure of methane with temperature. This is governed by the Clausius-Clayperon relation:

$$\frac{\mathrm{d}P_v}{\mathrm{d}T} = \frac{P_v L \mu_w}{T^2 R}. \tag{36}$$

Integrating, we get the relation

$$\ln \frac{P_v}{P_0} = \frac{L\mu_w}{R}\left(\frac{1}{T_b} - \frac{1}{T}\right). \tag{37}$$

where $P_0 = 1.48\times10^5$ Pa and $T_b = 114$ K represent the boiling point of methane. As the temperature decreases when the parcel of air adiabatically expands, the vapour pressure will decrease. This causes the saturated methane vapour to condense and drop out of the system, which carries away latent heat.

Suppose we have a parcel of air at temperature T and pressure P and height h containing one mole of gas in total. The proportion of methane vapour is $P_v(T)/P \ll 1$. Let us consider what happens when this parcel rises by a height $\mathrm{d}h$, cools to a temperature T' and expands to a pressure P'. The parcel must satisfy hydrostatic equilibrium:

$$P' - P = \rho g \mathrm{d}h = \frac{P\mu g \mathrm{d}h}{RT}, \tag{38}$$

where we neglect the contribution of the methane vapour to the mass of the parcel. The First Law of Thermodynamics tells us:

$$c_v\left(T' - T\right) = L\mu_w \Delta\left(\frac{P_v}{P}\right) - P\Delta\left(\frac{RT}{P}\right). \tag{39}$$

Converting these into differential equations:

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\frac{P\mu g}{RT} \tag{40}$$

$$\frac{5}{2}R\frac{\mathrm{d}T}{\mathrm{d}h} = -L\mu_w\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{P_v}{P}\right) - P\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{RT}{P}\right) \tag{41}$$

Simplifying and substituting:

$$\frac{7}{2}R\frac{\mathrm{d}T}{\mathrm{d}h} = \frac{RT}{P}\frac{\mathrm{d}P}{\mathrm{d}h} + L\mu_w P_v\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}h} - \frac{L\mu_w}{P}\frac{P_v L\mu_w}{T^2 R}\frac{\mathrm{d}T}{\mathrm{d}h} \tag{42}$$

$$\frac{7}{2}\frac{\mathrm{d}T}{\mathrm{d}h} = -\frac{\mu g}{R} - \frac{L\mu_w \mu g P_v}{PR^2T} - \frac{P_v\mu_w^2 L^2}{PT^2R^2}\frac{\mathrm{d}T}{\mathrm{d}h} \tag{43}$$

$$\frac{\mathrm{d}T}{\mathrm{d}h} = \frac{-\mu g - L\mu_w\mu g P_v/PRT}{7R/2 + L^2\mu_w^2 P_v/PT^2R} \tag{44}$$

$$= -\frac{2\mu g}{7R}\frac{1 + L\mu_w P_v/PRT}{1 + 2L^2\mu_w^2 P_v/7PR^2T^2}. \tag{45}$$

To find the altitude at which clouds can no longer form, we want to find the altitude at which the air is sufficiently cooled to be effectively dry. To find this, we first consider the simplifying assumption that $T = T_0 - \Gamma h$, i.e. the temperature has a constant lapse rate. Substituting into (37) we get

$$\ln \frac{P}{P_0} = \frac{\mu g}{R\Gamma} \ln \frac{T_0 - \Gamma h}{T_0} \tag{46}$$

$$P = P_0 \left(1 - \frac{\Gamma h}{T_0}\right)^{\mu g/R\Gamma}. \tag{47}$$

The initial value of the lapse rate is $\Gamma = 3.960 \times 10^{-3}\,\mathrm{K/m}$. We estimate the evolution of T by iterating this approximation over intervals of h and setting new initial conditions to be the values calculated by the last step. (This is essentially a slightly scuffed version of a finite difference method.)

h	T	P	P_v	$-\frac{dT}{dh}$	$\frac{P_v}{P}$
1000	296.0	8.92×10^4	3296	4.052×10^{-3}	3.70×10^{-2}
2000	291.9	7.94×10^4	2613	4.165×10^{-3}	3.29×10^{-2}
3000	287.7	7.06×10^4	2046	4.299×10^{-3}	2.90×10^{-2}
4000	283.4	6.26×10^4	1581	4.456×10^{-3}	2.53×10^{-2}
5000	278.9	5.54×10^4	1196	4.651×10^{-3}	2.16×10^{-2}
6000	274.2	4.90×10^4	885.6	4.890×10^{-3}	1.81×10^{-2}
7000	269.3	4.32×10^4	640.1	5.175×10^{-3}	1.48×10^{-2}
8000	264.1	3.80×10^4	447.6	5.526×10^{-3}	1.18×10^{-2}
9000	258.8	3.33×10^4	306.3	5.918×10^{-3}	9.20×10^{-3}
10000	252.9	2.91×10^4	197.0	6.413×10^{-3}	6.76×10^{-3}
11000	246.5	2.54×10^4	119.2	6.993×10^{-3}	4.69×10^{-3}
12000	239.5	2.21×10^4	66.74	7.631×10^{-3}	3.02×10^{-3}
13000	231.9	1.91×10^4	34.17	8.265×10^{-3}	1.79×10^{-3}
14000	223.6	1.64×10^4	15.61	8.830×10^{-3}	9.52×10^{-4}
15000	214.8	1.40×10^4	6.370	9.253×10^{-3}	4.55×10^{-4}

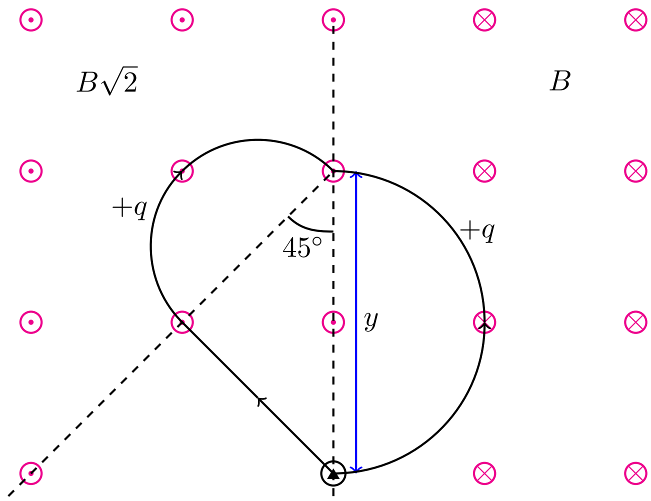
Table 1: Iterating for the values of various physical parameters over different values of h.

One can see that the water vapour content drops effectively to 0 (and the lapse rate approaches the dry lapse rate) at approximately $h = \boxed{15\,\mathrm{km}}$. (This is verifiable by numerical simulation; due to the approximate nature of the manual method and the cloud formation threshold, answers from 12 km (where the water vapour content drops to below 10% that of the starting water vapour content) to 16 km should be accepted.)

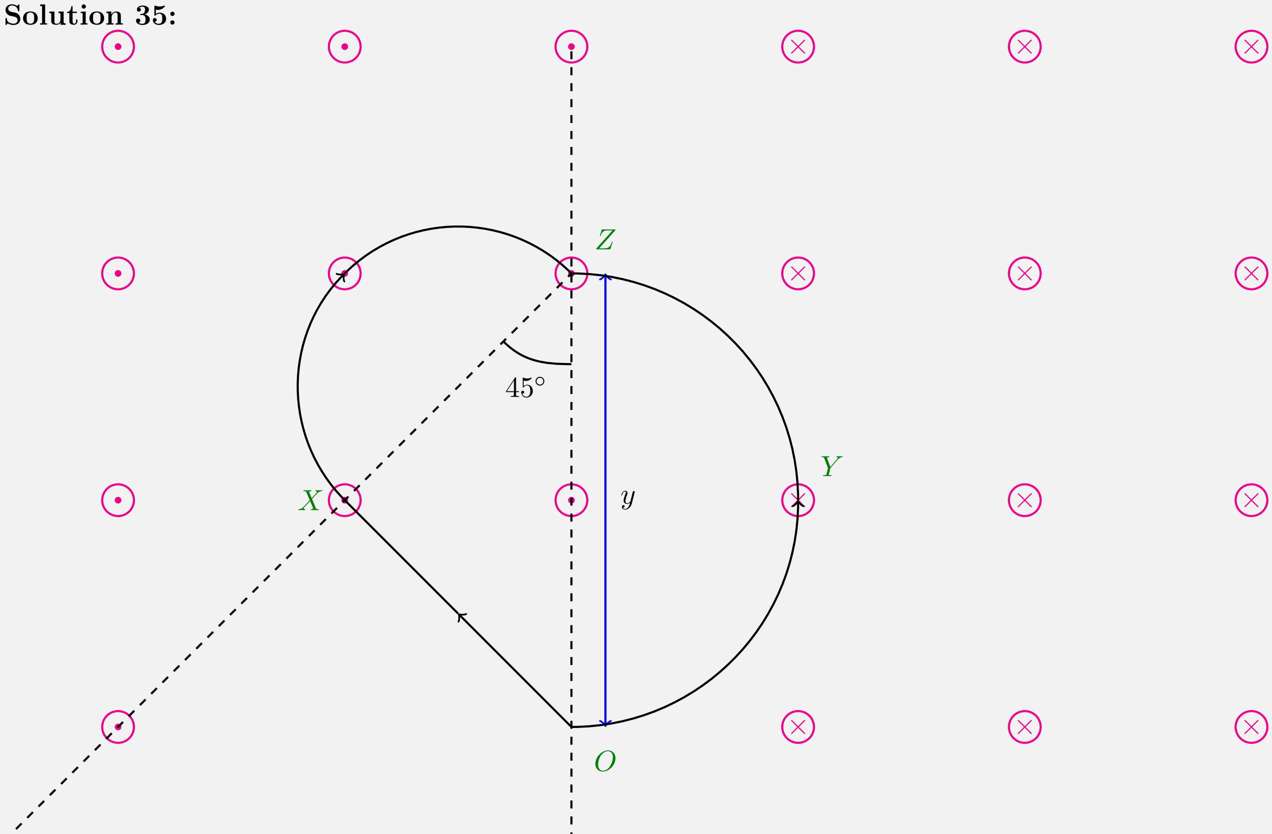
35. Airplane, QED

In their quest for world domination, the OPhO committee has created an airplane beam generator (ABG) which produces two beams of positively charged airplanes. These airplanes have mass $m = m_e$, where m_e is the mass of an electron, and are produced at speed $v = 1.00 \cdot 10^{-3}c$. The two beams are offset by an angle of 135°. The left facing beam is positioned at 45° above the horizontal and hence the right facing beam is parallel to the horizontal.

They mount the ABG on a rail which is mounted parallel to the vertical, and can hence move up and down freely. They set up a magnetic field such that in the 2nd quadrant and the top left half (from the horizontal to 45° down from the horizontal) of the 3rd quadrant, the field is $B\sqrt{2}$ **out of** the page, and in the 1st and 4th quadrant, the field is $B = 1$ T **into** the page. The ABG is then moved such that the two beams intersect at the origin, and there is a phase difference of exactly 2π between the beams at the origin. What must be the charge q on the airplane for this to occur? Assume the airplane source is coherent and that there is no initial phase difference. Make sure to submit $|q|$.



Solution 35:



The phase difference is given by $\Delta\varphi = (1/\hbar)\int \mathbf{p}\cdot \mathrm{d}\mathbf{x}$. The Lagrangian of a free particle in an EM field is given by $\mathcal{L} = m\mathbf{v}^2/2 + q\mathbf{A}\cdot\mathbf{v} - q\phi$. Hence the generalised momentum is $\partial\mathcal{L}/\partial\mathbf{v} = m\mathbf{v} + q\mathbf{A}$.

To find $\mathbf{A}$, note that $\mathbf{B} = \nabla \times \mathbf{A}$, and therefore by Stokes' theorem

$$\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \iint (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \iint \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

In a uniform magnetic field, charged particles undergo circular motion as $r = mv/qB$. Let $y = 2r$. Then,

$$\begin{aligned}
\Delta\varphi = \pm 2\pi &= \frac{1}{\hbar}\left[p\left(2r\sin\frac{\pi}{4} + \pi r\cos\frac{\pi}{4}\right) + q\int_{OXZ}\mathbf{A}\cdot\mathrm{d}\mathbf{l} - p(\pi r) - \int_{OYZ}\mathbf{A}\cdot\mathrm{d}\mathbf{l}\right] \\
\pm 2\pi &= \frac{1}{\hbar}\left[p\left(r\sqrt{2} + \pi\frac{r}{\sqrt{2}} - \pi r\right) + q\int_{OXZ}\mathbf{A}\cdot\mathrm{d}\mathbf{l} + q\int_{ZYO}\mathbf{A}\cdot\mathrm{d}\mathbf{l}\right] \\
\pm 2\pi &= \frac{1}{\hbar}\left[pr\left(\sqrt{2} + \pi\left(\frac{1}{\sqrt{2}} - 1\right)\right) + q\oint_{OXZYO}\mathbf{A}\cdot\mathrm{d}\mathbf{l}\right] \\
\pm 2\pi &= \frac{1}{\hbar}\left[pr\left(\sqrt{2} + \pi\left(\frac{1}{\sqrt{2}} - 1\right)\right) + q\iint\mathbf{B}\cdot\mathrm{d}\mathbf{S}\right].
\end{aligned}$$

Since $OXZYO$ is clockwise, $\mathrm{d}\mathbf{S}$ is into the page. Also note that $r = p/qB$ and so $p = qBr$. For simplicity, let $k \equiv \sqrt{2} + \pi\left(1/\sqrt{2} - 1\right)$. Hence,

$$\begin{aligned}
\pm 2\pi &= \frac{1}{\hbar}\left[qBr^2k - \frac{1}{2}q\pi\left(B\sqrt{2}\cdot\left(\frac{r}{\sqrt{2}}\right)^2 - Br^2\right)\right] \\
&= \frac{1}{\hbar}qBr^2\left(k - \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2}\right) \\
&= \frac{m_e^2v^2}{\hbar}\frac{1}{qB}\left(k - \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2}\right) \\
q &= \frac{m_e^2v^2}{\hbar B}\frac{\sqrt{2} + \pi/2\sqrt{2} - \pi/2}{2\pi},
\end{aligned}$$

where we have taken the -ve value of 2π since $q > 0$ (note that this doesn't actually change anything since $|q|$ remains the same). Then numerically, $q = \boxed{1.075 \times 10^{-16}\ \mathrm{C}}$.

Remark: The original version of this problem asked for the phase difference between a beam of particles and a beam of antiparticles. Can you figure out why this is unphysical?