2025 Online Physics Olympiad: Invitational Contest

Part I

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Instructions

Part 1 of the Invitational Round consists of three theoretical free response questions over 2 full days from August 23, 12:00 AM UTC to August 24, 11:59 PM UTC. Part 2 consists of a fourth theoretical question and one experimental problem, to be completed during the second day, from August 24, 12:00 AM UTC to August 24, 11:59 PM UTC.

- **The team leader must submit their final solution document in this google form. Only your last submission to the form will be graded. Submissions by any other means will not be graded.** We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it. You are allowed to submit up 10 MB of data for each problem solution. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Since each question is a long answer response, participants will be judged on the quality of their work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

- **Participants are required to keep all their rough working for both parts of this round and to submit it to this form by 30 minutes after the end of the competition, i.e. the deadline is August 25, 12:30 AM UTC. Teams that do not submit rough work will be disqualified. Only your last submission to the form will be considered.** Note that we will not request working from any specific team: all teams are required to submit their rough working.

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^{8}$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\mathrm{N} \cdot \mathrm{m}^2)/\mathrm{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \mathrm{kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N \cdot m^2})/\mathrm{C^2}$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\mathrm{T \cdot m})/\mathrm{A}$$
- 1 atmospheric pressure,
$$1\ \mathrm{atm} = 1.01 \cdot 10^{5}\ \mathrm{N/m^2} = 1.01 \cdot 10^{5}\ \mathrm{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$$

Theory Question 1 [10 marks]

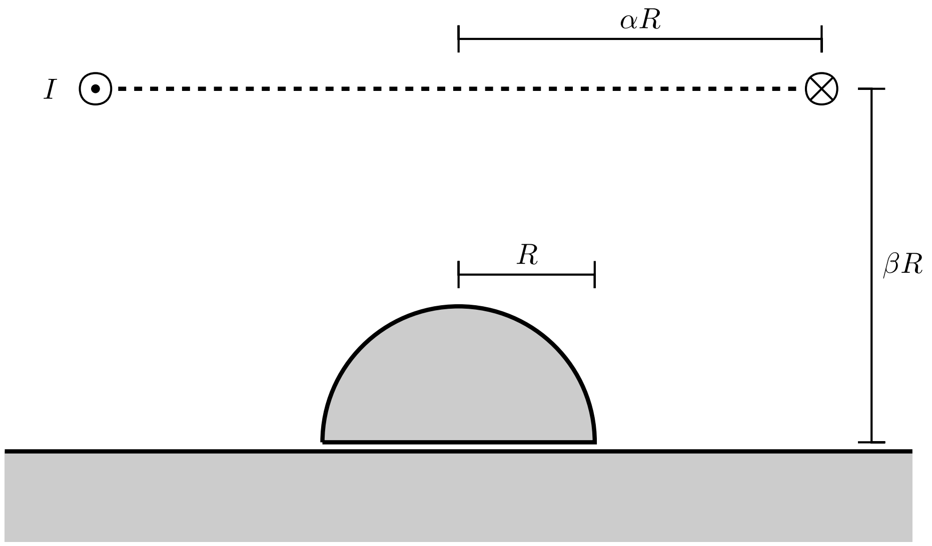
This problem consists of two independent parts.

A. Magnetosphere [5 marks]

Iron is a *soft ferromagnetic* material, meaning that it has relative permeability $\mu_r \gg 1$ and zero intrinsic magnetisation.

(A.1) Show that the magnetic field at the surface of an iron body is approximately perpendicular to the surface. [0.5 pts]

An iron hemisphere of radius R sits face down on a very large iron plane. Assume that there is a small air gap between the plane and the bottom of the hemisphere (with relative permeability 1). A thin conducting loop of radius αR is centered a distance βR above the hemisphere and has current I flowing through it.



(A.2) Find the net magnetic force on the hemisphere. [3 pts] *Give your answer in terms of the integral*

$$J(k) = \int_0^{2\pi} \frac{\cos(\phi)}{(k - \cos(\phi))^{3/2}}\, d\phi.$$

(A.3) Find numerically the maximum value of α such that the hemisphere can be lifted, assuming that β and I can be chosen freely. [0.5 pts]

(A.4) Find an equation for the maximum value of β such that the hemisphere can be lifted, assuming that α and I can be chosen freely. [1 pt]

B. World Eater [5 marks]

According to general relativity, a particle's distance r from a black hole of mass M obeys:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2} + \frac{\ell^2}{r^3} - \frac{3GM\ell^2}{c^2 r^4}$$

Here, τ is the particle's proper time and ℓ is its conserved angular momentum per unit mass around the black hole.

A black hole of mass M travels at speed v through a medium with density ρ, accreting mass as it moves. Near the black hole, the gravitational field is so strong that you may treat the medium as a fluid of noninteracting particles. Assume that $(\rho G^3 M^2)^{1/6} \ll v \ll c$.

(B.1) Find the rate of increase of the black hole's mass. [1.5]

(B.2) Find the drag force on the black hole, neglecting the medium's gravitational interaction with itself. Qualitatively explain your result. [2]

(B.3) Including the medium's gravitational interaction with itself, give an asymptotic estimate for the drag force. [1.5]

Magnetosphere - Solution

(a) Consider the fields $\vec{B}_0, \vec{H}_0$ and $\vec{B}_1, \vec{H}_1$ outside and inside the iron respectively. Gauss's Law, $\nabla \cdot \vec{B} = 0$, shows that $B_{0,\perp} = B_{1,\perp}$ while Ampere's Law, $\nabla \times \vec{H} = \vec{J}_f = \vec{0}$, shows that $\vec{H}_{0,\parallel} = \vec{H}_{1,\parallel}$. These are the boundary conditions for magnetic materials. Using $\vec{H} = \vec{B}/\mu$, we have $\vec{B}_{1,\parallel} = \mu_r \vec{B}_{0,\parallel}$. Thus, if θ_0 and θ_1 are the angles to the normal, we have:

$$\tan(\theta_0) = \frac{B_{0,\perp}}{|\vec{B}_{0,\parallel}|},\ \tan(\theta_1) = \frac{B_{1,\perp}}{|\vec{B}_{1,\parallel}|} \implies \frac{\tan(\theta_1)}{\tan(\theta_0)} = \mu_r$$

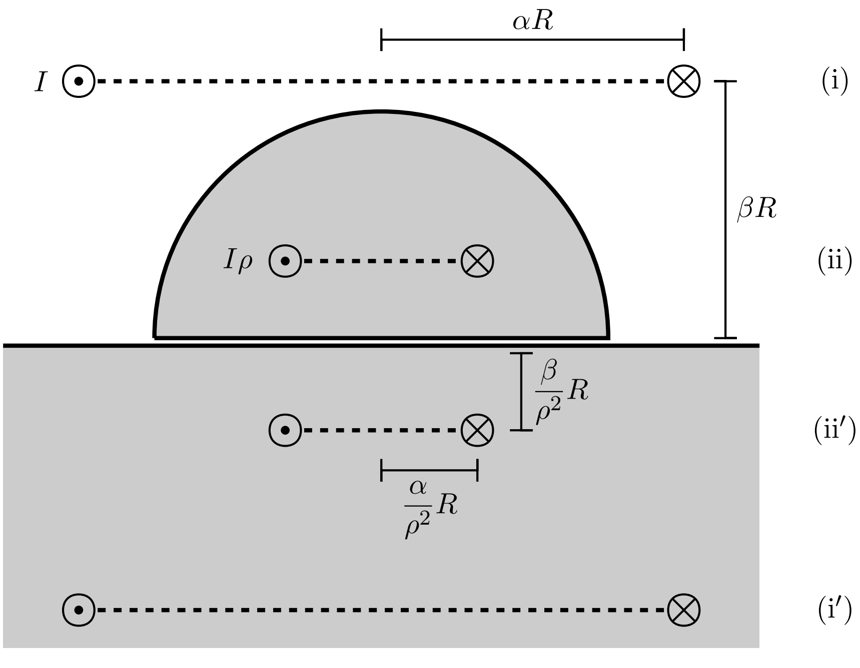
Assuming that $B_{1,\parallel}$ cannot become arbitrarily large, this implies that $\theta_0 \to 0$ as $\mu_r \to \infty$.

Marking Scheme: 0.25 for applying $\vec{B}$ and $\vec{H}$ boundary conditions. 0.25 for obtaining "refraction" law relating θ_1 and θ_2.

Original solution for (b)-(d) (b) By uniqueness, it suffices to find a field $\vec{B}$ that is perpendicular to the interfaces and has the correct circulation around the wire, so we may use the method of images. Note that the image of the current loop in the plane is a congruent current loop with the same orientation.

To show that the image of the current loop in the sphere is another current loop, consider the equivalent configuration of a spherical cap with radius αR and magnetic dipole moment per unit area I perpendicular to the cap. This lets us apply well-known results from electrostatics. The image of this cap is another cap with radius $\alpha' R = \frac{\alpha}{\rho^2} R$, where $\rho^2 = \alpha^2 + \beta^2$. Furthermore, the image of an area dA has area $dA' = \frac{dA}{\rho^4}$ and dipole moment $dm' = \frac{dm}{\rho^3}$, giving $I' = \frac{dm'}{dA'} = I\rho$ (with the same orientation).

Thus, we can consider the following set of three image loops. Because the loops can be divided into image pairs either in the sphere or in the plane, the resulting field is perpendicular to both of these surfaces and thus satisfies the boundary conditions.



Next, we note that the force on the hemisphere is equal to the force on loop (ii) from the other three. This is because we could replace the hemisphere with loop (ii) without changing the external field, so the forces on loop (i) and on the plane would not change. Here, it is necessary to have an air gap between the hemisphere and the plane.

Thus, we need to find the force between coaxial current loops. Consider the general case of two loops with radii R, aR and currents I_1, I_2 whose centers are at $z = 0, bR$. First, we'll find the magnetic vector potential of the first loop, working in cylindrical coordinates:

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I_1}{4\pi}\int \frac{d\vec{\ell}}{r} \\ &= \frac{\mu_0 I_1}{4\pi}\int_0^{2\pi} \frac{R(-\sin(\phi)\hat{r} + \cos(\phi)\hat{\phi})}{\sqrt{R^2 + (aR)^2 + (bR)^2 - 2R(aR)\cos(\phi)}}\, d\phi \\ &= \frac{\mu_0 I_1}{4\pi}\int_0^{2\pi} \frac{\cos(\phi)}{\sqrt{1 + a^2 + b^2 - 2a\cos(\phi)}}\, d\phi\ \hat{\phi}\end{aligned}$$

Here, the $\hat{r}$ component integrates to 0. Thus, the force on the second loop is:

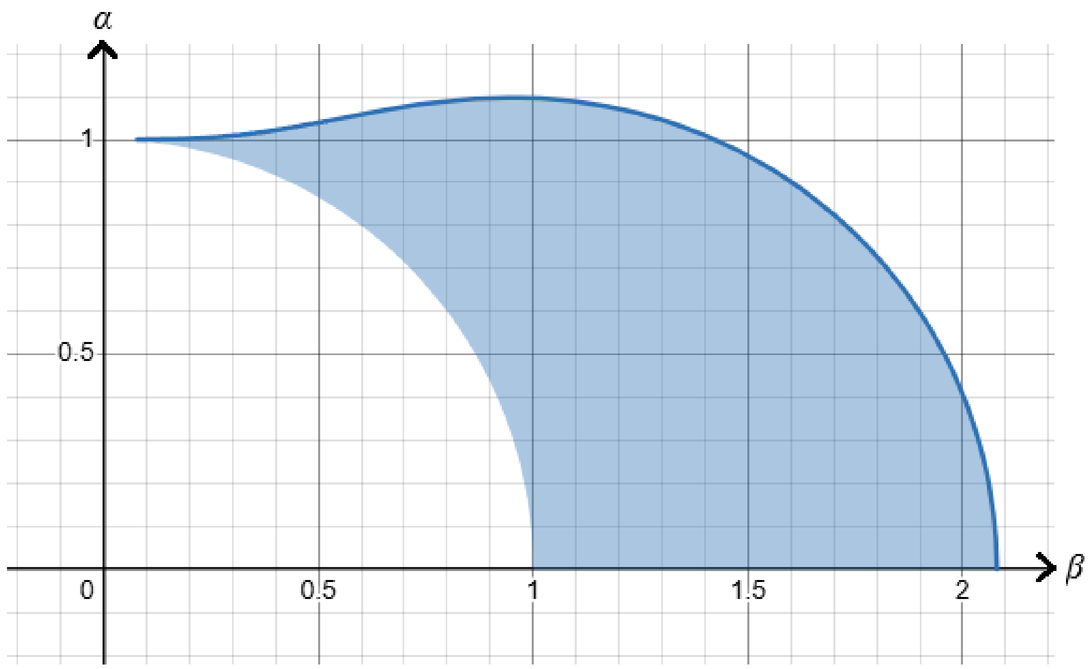
$$\begin{aligned}F_z(a, b, I_1, I_2) &= -2\pi(aR)I_2 B_r \\ &= -2\pi a I_2 R\left(-\frac{\partial A_\phi}{\partial z}\right) \\ &= 2\pi a I_2 R\cdot\frac{\mu_0 I_1}{4\pi}\int_0^{2\pi} \frac{1}{R}\frac{d}{db}\left(\frac{\cos(\phi)}{\sqrt{1 + a^2 + b^2 - 2a\cos(\phi)}}\right) d\phi \\ &= -\frac{\mu_0 I_1 I_2 b}{4\sqrt{2a}} J\left(\frac{1 + a^2 + b^2}{2a}\right)\end{aligned}$$

Note that this is independent of R. Finally, we can write the total force on loop (ii) as:

$$F_{(\text{ii})} = F_z\left(\frac{1}{\rho^2}, \frac{\beta}{\alpha}\left(\frac{1}{\rho^2} - 1\right), I, I\rho\right) + F_z\left(\frac{1}{\rho^2}, \frac{\beta}{\alpha}\left(\frac{1}{\rho^2} + 1\right), I, I\rho\right) + F_z\left(1, \frac{2\beta}{\alpha}, I\rho, I\rho\right)$$

$$= \boxed{\frac{\mu_0 I^2 \beta}{4\sqrt{2}\alpha}\left((\rho^2 - 1)J\left(\frac{\rho^4 - 2\beta^2 + 1}{2\alpha^2}\right) - (\rho^2 + 1)J\left(\frac{\rho^4 + 2\beta^2 + 1}{2\alpha^2}\right) - 2\rho^2 J\left(1 + \frac{2\beta^2}{\alpha^2}\right)\right)}$$

(c) Plotting the region where $F_{(\text{ii})} > 0$ in the (α, β) plane:



We see that the maximum value of α is $\boxed{\alpha \approx 1.097}$; here, $\beta \approx 0.948$.

(d) According to the plot, the maximum value of β has $R = 0$, so we can treat the current rings as point dipoles. The field of a dipole m along its axis is $B_z = \dfrac{\mu_0 m}{2\pi z^3}$, so the force on a dipole m' located at $z = z_0$ is $F_z = m' \dfrac{dB_z}{dz} = -\dfrac{3\mu_0 m m'}{2\pi z_0^4}$. Letting ring (i) have moment m, ring (i′) has moment m while rings (ii) and (ii′) have moments $\dfrac{m}{\beta^3}$. Thus, at the threshold we have:

$$F_{(\text{ii})} = \frac{3\mu_0 m^2}{2\pi R^4}\left(\frac{1/\beta^3}{(\beta - 1/\beta)^4} - \frac{1/\beta^3}{(\beta + 1/\beta)^4} - \frac{(1/\beta^3)^2}{(2/\beta)^4}\right) = 0$$

$$\Rightarrow \boxed{\frac{1}{(\beta^2 - 1)^4} - \frac{1}{(\beta^2 + 1)^4} - \frac{1}{16\beta^3} = 0}, \quad \beta \approx 2.079$$

This agrees with the plot.

Erratum Unfortunately, the image charges for part (b) are in error. The image of a dipole in a sphere is not another dipole; instead, it is a dipole and a net charge. If we think of the dipole as two opposite charges, the image charges are no longer of equal magnitude, and the difference is non-negligible for any choice of charges. This is a subtle issue which is detailed further in Kevin Zhou's E2 handout.

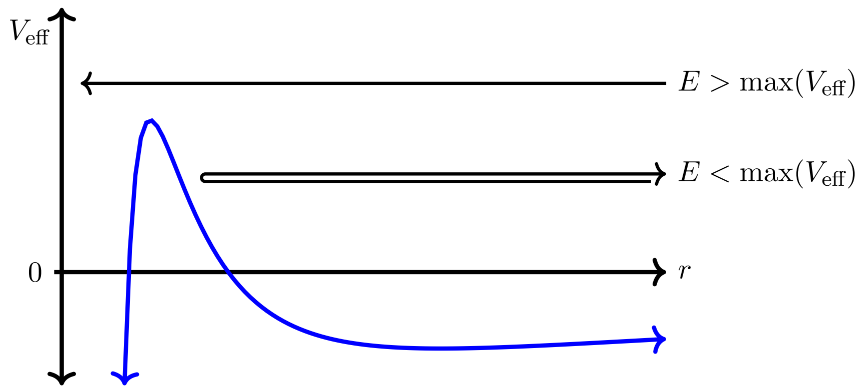
For the case of a ferromagnetic sphere, the net flux out of the sphere must be 0 by Gauss' law, so we add another charge at the center of the sphere to enforce neutrality. Then, the image of a current ring can be thought of as another current ring and a conical sheet of current from the image ring to the center of the sphere. This situation is much less tractable than intended, so we have decided to give full points for (b)-(d) to teams which realized the key idea of the sphere-plane image charge setup.

World Eater – Solution

(a) Defining $\alpha = \frac{rc^2}{GM}, \beta = \frac{\ell c}{GM}$, we have the following:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{dV_{\text{eff}}}{dr}, \quad V_{\text{eff}} = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3} = c^2\left(-\frac{1}{\alpha} + \frac{\beta^2}{2\alpha^2} - \frac{\beta^2}{\alpha^3}\right)$$

Thus, r evolves under the "effective potential" V_{eff}. For $r \to \infty$, we have $V_{\text{eff}} \to 0$, so a faraway particle can overcome the angular momentum barrier and fall into the black hole if its "effective energy" $E = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2$ satisfies $E > \max(V_{\text{eff}})$.



Consider a particle with impact parameter (original distance to the axis of motion) b. Then, we have $\ell \approx bv$ and $E \approx v^2/2$. Let b_m be the maximum impact parameter which falls into the black hole. We claim that $b_m \approx b_0$ where b_0 gives $\max(V_{\text{eff}}) = 0$. To see this, let β_m, β_0 be the corresponding values of β. Then, we can linearly approximate $\max(V_{\text{eff}}) \sim c^2(\beta_m - \beta_0)$, so $\max(V_{\text{eff}}) = E$ gives $\beta_m = \beta_0 + O(v^2/c^2) \approx \beta_0$.

Note that when $\max(V_{\text{eff}}) = 0$, the equation $V_{\text{eff}}(\alpha) = 0$ will have a double root. We have:

$$V_{\text{eff}}(\alpha) = c^2\left(-\frac{1}{\alpha} + \frac{\beta_0^2}{2\alpha^2} - \frac{\beta_0^2}{\alpha^3}\right) = 0 \Rightarrow \alpha^2 - \frac{\beta_0^2}{2}\alpha + \beta_0^2 = 0$$

This has a double root when $\beta_0 = 4$, so $b_m \approx b_0 = \frac{4GM}{cv}$. Finally, the rate of mass increase is equal to the rate of mass swept out by the circle of infalling impact parameters, which is $\pi\rho v b_m^2 \approx \boxed{\frac{16\pi\rho G^2 M^2}{c^2 v}}$.

Marking Scheme: 0.25 for rewriting in terms of effective potential. 0.5 for considering particles with a given impact parameter and realizing that they will enter the black hole iff they can overcome the effective potential. 0.5 for finding the maximum impact parameter in the $v \ll c$ approximation. 0.25 for finding the rate of mass increase using the swept out circle of impact parameters.

(b) Treat the black hole's potential as Newtonian. Relativistic behavior is important for $b \sim \frac{GM}{cv}$, so we may neglect it because the Newtonian scale is $b \sim \frac{GM}{v^2}$ and $v \ll c$.

Note that all particles leave with speed v in the black hole's frame, so we need to find the deflection angle $\phi(b)$. Using the properties of a hyperbola, we find $\phi = -2\arctan(a/b)$ where $a < 0$ is the semimajor axis. We have $E = -\frac{GMm}{2a}$, so $a = -\frac{GM}{v^2}$. Then, the change in the particle's axial momentum is:

$$dp_{\parallel} = mv(1-\cos(\phi)) = 2mv\sin^2\left(\arctan\left(\frac{a}{b}\right)\right) = \frac{2mv}{1+(bv^2/GM)^2}$$

Integrating over all particles:

$$F = \frac{dp_{\parallel}}{dt} = \int \frac{2v}{1+(bv^2/GM)^2}\,\frac{dm}{dt} = \int_0^{\infty} \frac{2v}{1+(bv^2/GM)^2}\cdot 2\pi b\rho v\; db$$

Letting $u = 1+(bv^2/GM)^2$:

$$F = \frac{2\pi\rho G^2M^2}{v^2}\int_1^{\infty}\frac{1}{u}\,du$$

This $\boxed{\text{diverges}}$! Clearly, the assumption that particles arbitrarily far away are affected only by the black hole is not valid.

Marking Scheme: 0.25 for stating that relativistic trajectories can be neglected. 0.75 for finding the axial impulse from a particle starting at a given impact parameter in the Newtonian approximation. Small deflections end up being the dominant contribution, so it is valid to approximate particles' trajectories as straight lines. 0.5 for integrating over all impact parameters. 0.25 for obtaining a logarithmic divergence and 0.25 for the correct prefactor.

(c) We must apply a cutoff scale $b < b_m$ to the integral. This is the distance where the gravitational influence of the black hole no longer overwhelms that of the medium; dimensionally, we have $b_m \sim (M/\rho)^{1/3}$. Then, $u_m \approx \left(b_m v^2/GM\right)^2 \sim \left(\frac{v^6}{\rho G^3M^2}\right)^{2/3}$; because $(\rho G^3M^2)^{1/6} \ll v$, we have $u_m \gg 1$. This gives:

$$F = \frac{2\pi\rho G^2M^2}{v^2}\ln(u_m) \approx \boxed{\frac{4\pi\rho G^2M^2}{3v^2}\ln\left(\frac{v^6}{\rho G^3M^2}\right)}$$

Note that a multiplicative uncertainty in b_m turns into an additive uncertainty in the logarithm, allowing us to obtain an asymptotic estimate.

Marking Scheme: 1.0 for realizing that a cutoff scale is needed due to assumptions about the noninteracting medium becoming invalid. Different cutoff scales (i.e. the dimensional scale presented here, or a different scale based on the Jeans' length of the black hole) may lead to different numeric prefactors; we accepted any which give the correct logarithmic term for 0.5 points.

Theory Question 2 [10 marks]

This problem consists of two independent parts.

A. Efficient Lighting [5.5 marks]

Let S be a two-dimensional, simply connected room with perfectly reflecting walls, illuminated by omnidirectional light sources located in the room's interior. We strictly work in the geometric optics regime and neglect all reflections off of non-differentiable corners. Sources do not affect light passing through them.

(A.1) For any given configuration of $n \geq 2$ distinct point sources, construct S such that the parts of the room illuminated by each source are disjoint, and the total illuminated area is equal to the area of S. Sketch two concrete examples of your construction that differ in more than a trivial sense. [1 pt]

(A.2) Now, for a finite number $n \geq 2$ sources which are finite, disjoint and simply connected 2D regions, whose shape and placement are **of your choice**, construct S satisfying the conditions of part (a). Sketch your construction of S and indicate the chosen source shape and placement. [2 pts]

(A.3) Likewise, for any **arbitrary** $n \geq 2$ sources which are finite, disjoint and simply connected 2D regions, describe how to generally construct S satisfying the conditions of part (a). Sketch one possible example of your construction, differing in more than a trivial sense from your answer to part (b). [2.5 pts]

B. Gas Gas Gas [4.5 marks]

(B.1) Consider a viscous gas cloud of finite extent orbiting around a dominating central mass. Prove that no steady state or periodic solution exists. Do not assume any specific equation of state (i.e., relation between P and ρ). [4.5 pts] *Note: Solutions which do not demonstrate the non-existence of local minima will receive no points. Solutions which prove the result in the case of a disk will receive half points.*

Efficient Lighting - Solution

Ivan's solution

(A.1) The problem might be reduced to considering the room S as a sequence of chambers, each containing one light source, such that all of the light rays emitted from the source stay confined within the corresponding chamber. One may be inspired by the fact that, considering a circle with a light source at its centre, all light rays arrive at the centre after exactly one reflection off the circular wall. The radius of the circle being irrelevant regarding reflections, one can select specific parts of the wall and prolong said parts, obtaining the room illustrated by Figure (1). Note that the light sources **do not** necessarily need to be collinear.

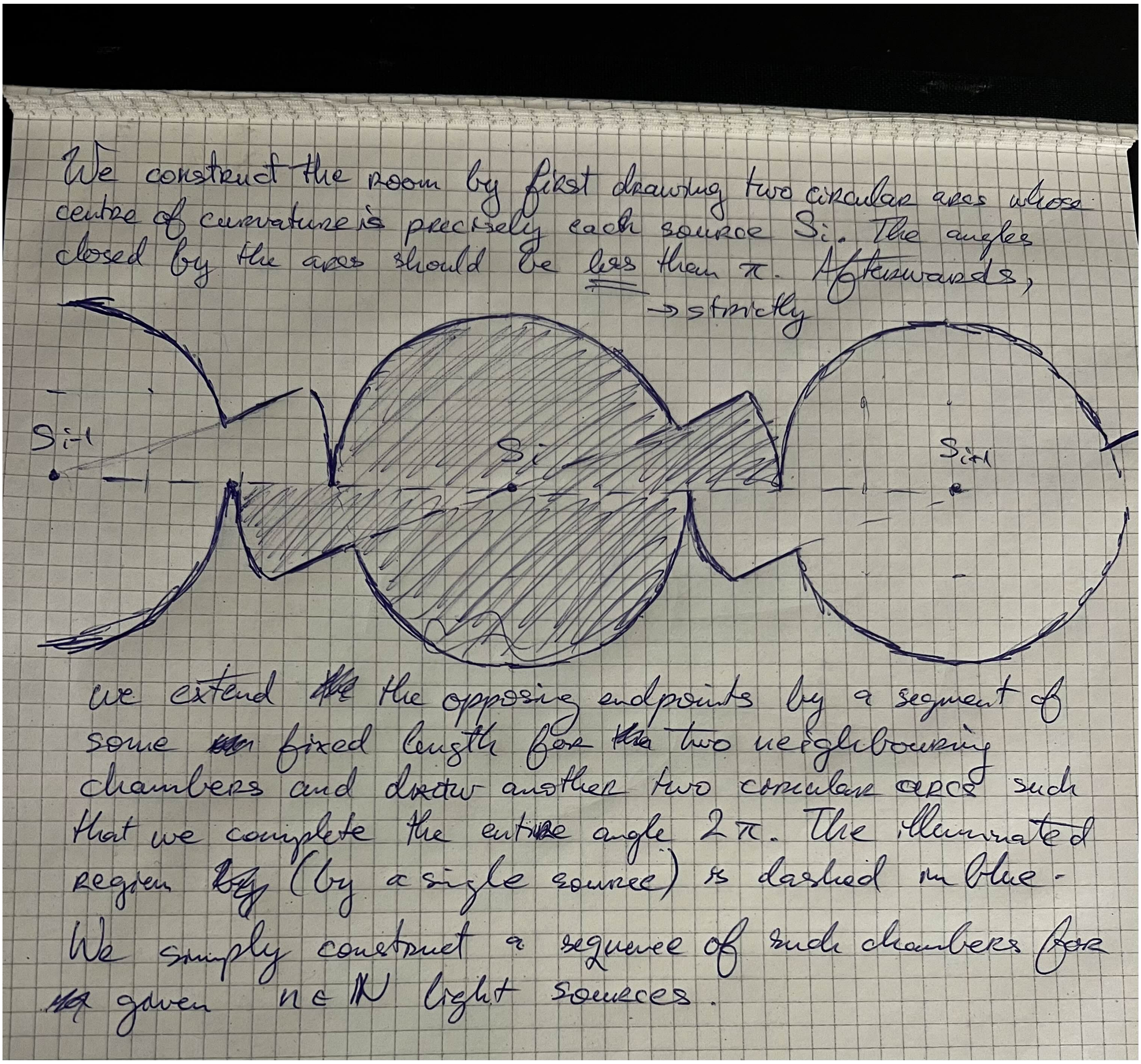


Figure 1: Possible first construction for (A.1)

Another possible solution is to consider the reflective properties of parabolas. Construct chambers around each source made of parabolic walls such that one is wider than the other. The gaps between them allow only strictly vertical light rays to pass, enabling the construction of the entire room S by prolonging these corridors in any way. Note the 45° angle for corridors that need to bend to reach another light source.

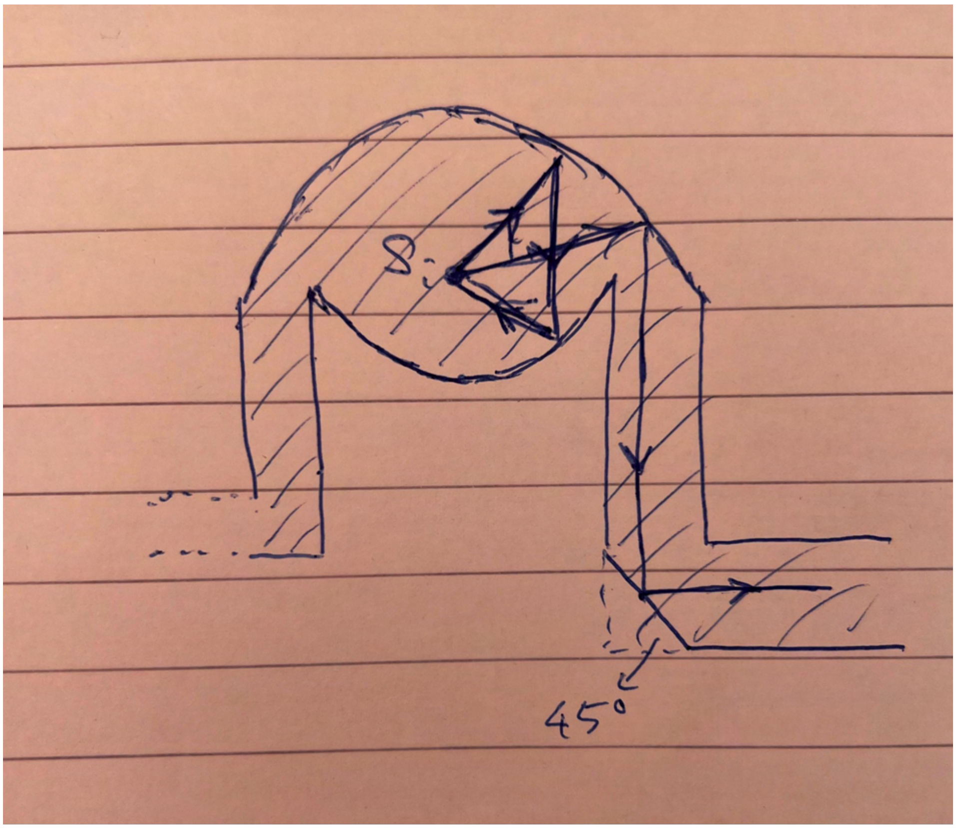


Figure 2: Possible second construction for (A.1)

Another interesting room to consider is the following one, which uses elliptic and hyperbolic walls. However, in this case, the light sources need to be strictly collinear.

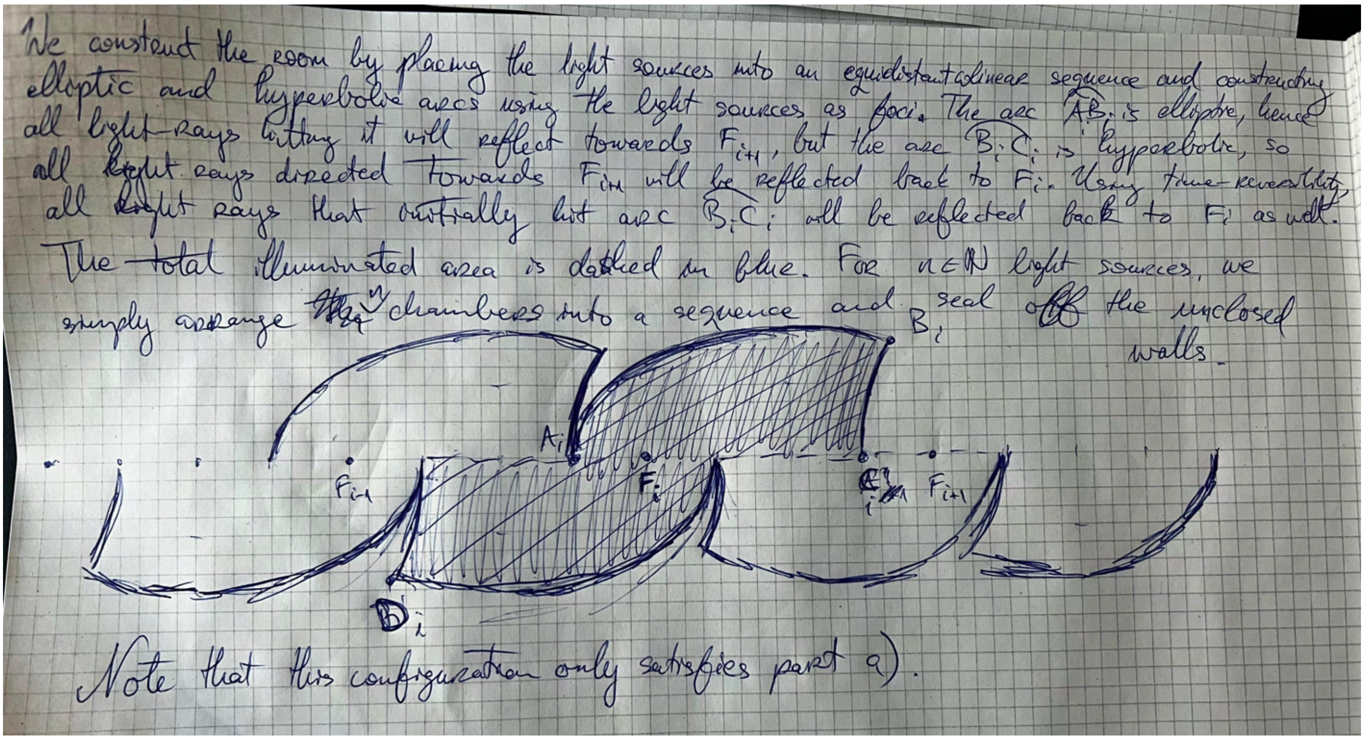


Figure 3: Interesting construction, but not a solution to any subpart.

(A.2) / (A.3) Analogous to part (a), this part is concerned with controlling all light rays emitted from within some region R to return to R after a finite number of reflections. This is easily achieved by considering the reflective property of an arbitrary ellipse, so that one may attempt the following construction of S:

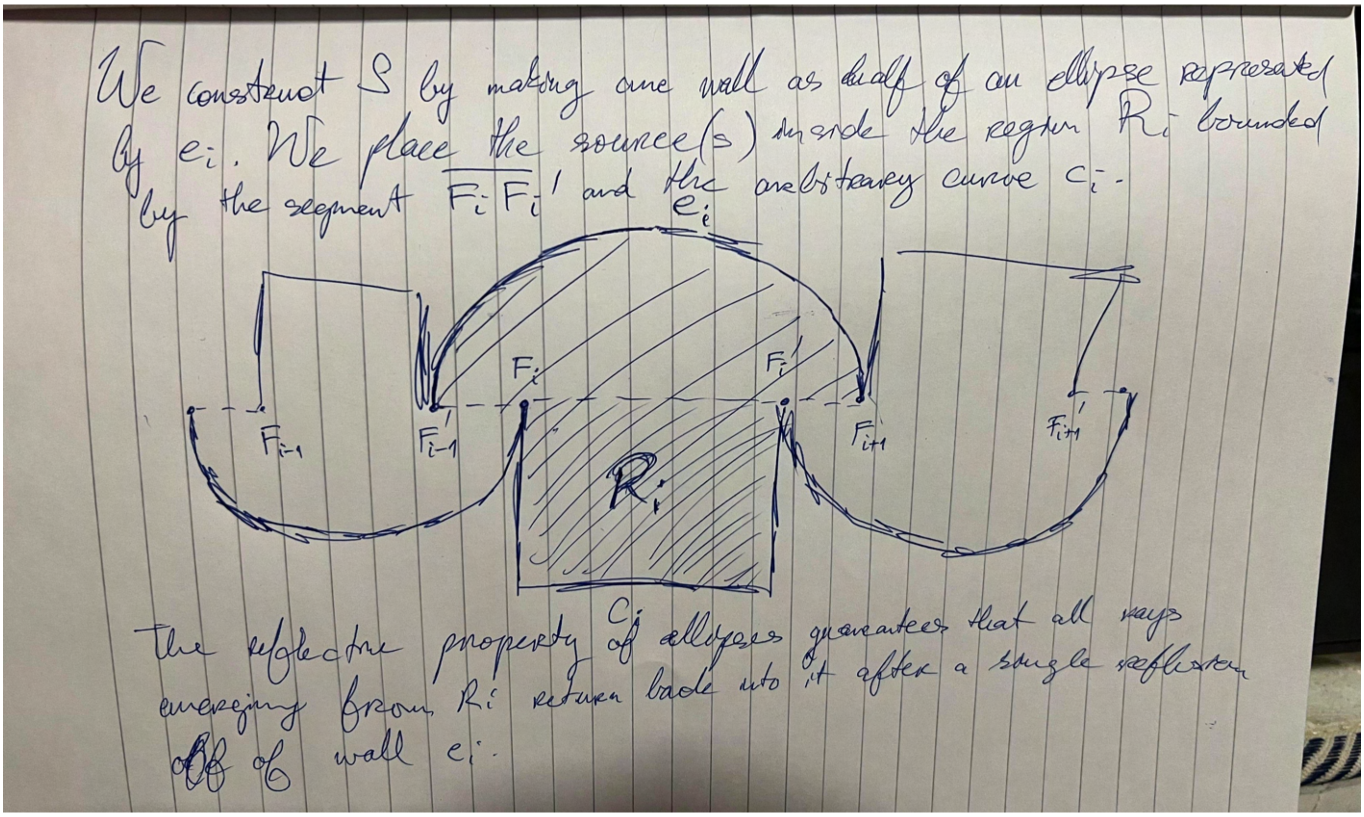


Figure 4: Possible solution for (A.3)

Note that the regions R_{i} are of arbitrary shape, hence one can build any corridor/canal towards a specific region-like source. The same can be obtained by considering parabolas:

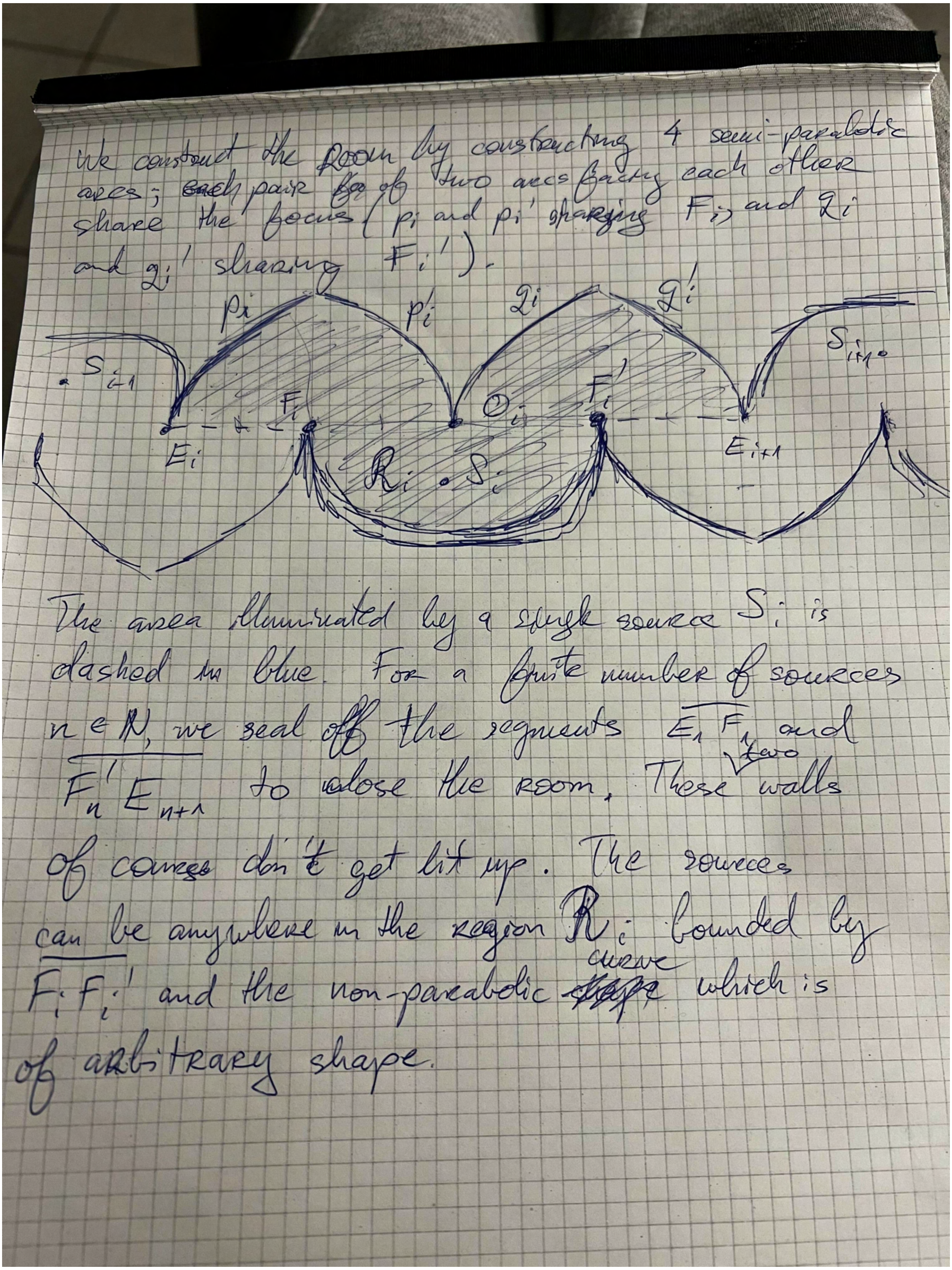


Figure 5: Possible solution for (A.3)

Hubert's solution: Formal proof for the existence of configurations for both parts follows.

(A.1) Suppose we have a finite set of point sources $S = \{x_i, i \in [1, n]\}$. We claim that it is possible to choose an ordering of i such that the line segments (x_i, x_{i+1}) form disjoint sets for all i (i.e. they do not intersect). This can be achieved by choosing an ordering such that the x-coordinates of x_i are arranged in an increasing order. This is achievable by the total orderedness of the reals. Should two points have the same x-coordinate we order points in order of increasing y-coordinate. Then, any two line segments have disjoint support on the x-axis and are therefore non-intersecting. For line segments which are vertical the same argument can be applied for support on the y-axis.

Given such an ordering and such a construction of line segments $l_i = (x_i, x_{i+1})$ we make the following construction: each point x_i is surrounded by the epsilon-ball $B_\epsilon(x_i)$. Then, along each line segment l_i for $i \in [1, n)$, construct two sectors which has a radius of $\frac{|x_i - x_{i+1}|}{2} + \delta_1$ (i.e. the sector reaches just beyond the midpoint of the line segment connecting the two points) centred on x_i and x_{i+1} respectively. The sectors should extend an angle δ_2 counterclockwise. We claim that the room consisting of the union of the epsilon-ball and all such sectors satisfies the requirements of the problem for some $\epsilon, \delta_1, \delta_2 > 0$.

Since a circle reflects all rays from its centre back towards the centre, it is clear that each union of the epsilon-ball and all sectors centred at x_i is fully illuminated by the source, and that all rays are parallel to any intersections of the boundaries with any source, so the only requirement is that these sets do not self intersect. In other words, if we consider that S_i is the union of $B_\epsilon(x_i)$ and the two sectors centred at x_i, we want to show that there exist $\epsilon, \delta_1, \delta_2 > 0$ such that $(S_i \backslash \partial S_i) \cap (S_j \backslash \partial S_j) = \emptyset$.

Firstly, we know the epsilon-balls do not intersect for some $\epsilon > 0$, since any finite set of points is nowhere dense. Suppose without loss of generality that the x-coordinates of x_i and x_{i+1} were different. (If not, the axes can be rotated infinitesimally anticlockwise.) Then we are able to choose δ_2 sufficiently small such that no point in the sector pointing towards x_{i+1} has an x-coordinate less than that of x_i by choosing it to be less than the slope of l_i. Assume further without loss of generality that x_{i-1} has a different x-coordinate with x_i (this is achievable for any configuration of three points). Then by choosing ϵ to be less than the finite difference of the x-coordinates of x_i and x_{i-1} we guarantee the non-intersection between the sector and the epsilon-ball around $B_\epsilon(x_{i-1})$. The intersection between two sectors occurs only on the boundaries by construction. Hence the condition is satisfied.

It is also easy to show that $\cup_i S_i$ is simply connected by observing that the curve consisting of the line segments l_i has no return path except through the same sequence of epsilon-balls and sectors, which can necessarily be contracted to a point. This completes the proof.

(A.2) / (A.3) Similar to the previous idea, we construct the room to be bounded by the boundary of each source, with a small corridor along the straight line connecting each source. The difference is that the incident angle of the light is arbitrary, so one needs to use an ellipse or parabola to refocus the light back towards the source, as shown in the illustration above. The $\epsilon\delta$ arguments will not be repeated as they are similar. The key point is to prove that, for any given finite disjoint sequence of simply connected sources S_i, we can choose an ordering for i and points $x_{i,2} \in \partial S_i$, $x_{i+1,1} \in \partial S_{i+1}$ such that the line segments $l_i = (x_{i,2}, x_{i+1,1})$ are pairwise disjoint with both $S_j \, \forall j$ and $l_j, j \neq i$.

This can be proven by construction via a greedy algorithm. For any pair of sets S_i, S_j we can define a minimum distance between the two sets, $D_{ij} = \min_{x_1 \in S_i, x_2 \in S_j} |x_1 - x_2|$. We find $\{i, j\} = \arg\min_{i \neq j} D_{ij}$. Then we find $\{x_1, x_2\} = \arg\min_{x_1 \in S_i, x_2 \in S_j} |x_1 - x_2|$ and construct the line segment (x_1, x_2). We then consider $S_i \cup S_j$ to be one set and reiterate the process until all sets are connected. The lines necessarily do not intersect with any set by the minimum distance property, and do not intersect with any other line as if they did, you can swap the pairs of points to obtain shorter lines, as in a minimum spanning tree. This proves the construction works for any arrangement of sources.

Gas - solution

We consider the gas cloud of density $\rho(\mathbf{x})$ and velocity field $\mathbf{v}(\mathbf{x})$. The total energy of this cloud is

$$E[\rho, \mathbf{v}] = \iiint_{\mathbb{R}^3} d^3\mathbf{x}\, \rho(-\frac{K}{r} + \mathbf{v}^2) \tag{1}$$

where we have neglected the constant factor, $r = |\mathbf{x}|$, and K is a constant relating to the mass of the central object. As viscous forces are always dissipative, we expect to minimise this functional. There are, however, several physics constraints, including the continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2}$$

as well as conservation of angular momentum, where we choose the units such that the magnitude is 1 and choose the z-axis to be along its direction (this is possible as all forces are internal):

$$\iiint_{\mathbb{R}^3} \rho \mathbf{x} \times \mathbf{v}\, d^3\mathbf{x} = \hat{\mathbf{z}} \tag{3}$$

We also note that the functional E is time-varying. We want to study the steady state, so we consider the time-average over an infinite horizon, which, as the limit tends to infinity, will be dominated by the steady state:

$$I[\rho, \mathbf{v}] = \lim_{T \to \infty} \frac{1}{T} \int_0^T E[\rho, \mathbf{v}] dt \tag{4}$$

We minimise this functional. With the two constraints, we introduce the Lagrange multipliers $\lambda(\mathbf{x}, t)$ and $\mu(t)$ for the continuity equation (which applies at every x and t) and the angular momentum conservation (which applies at every t). The full functional is therefore

$$I[\rho, \mathbf{v}] = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \iiint_{\mathbb{R}^3} d^3\mathbf{x}\, (\rho(-\frac{K}{r} + \mathbf{v}^2) - \mu(t) \cdot \rho \mathbf{x} \times \mathbf{v} - \lambda(\mathbf{x}, t)(\partial_t \rho + \nabla \cdot (\rho \mathbf{v}))) \tag{5}$$

Now, we vary this with respect to $\rho, \mathbf{v}, \lambda$ and μ, taking fixed T and taking the limit later. We also assume the boundary conditions that all quantities have finite support in $\mathbf{x}$ so boundary terms vanish (which is a physical assumption). The latter two return the constraints. Varying with respect to $\mathbf{v}$ gives

$$2\rho \mathbf{v} - \rho \nabla \lambda - \rho \mu \times \mathbf{x} = 0 \tag{6}$$

while varying over ρ gives

$$-\frac{K}{r} + \mathbf{v}^2 - \mu \cdot \mathbf{x} \times \mathbf{v} - \partial_t \lambda - \mathbf{v} \cdot \nabla \lambda = 0 \tag{7}$$

where in both cases we have implicitly taken the limit $\lambda \to 0$ as $t \to \infty$ as the steady state is unconstrained. Simplifying produces

$$2\mathbf{v} = \nabla \lambda + \mu \times \mathbf{x} \tag{8}$$

$$-\frac{K}{r} + \mathbf{v}^2 - \mu \cdot \mathbf{x} \times \mathbf{v} - \partial_t \lambda - \mathbf{v} \cdot \nabla \lambda = 0 \tag{9}$$

We take the dot product of both sides of (8) with $\mathbf{v}$ which gives

$$2\mathbf{v}^2 = \mathbf{v} \cdot \nabla \lambda + \mu \cdot \mathbf{x} \times \mathbf{v} \tag{10}$$

Substituting into (9) we get

$$\mathbf{v}^2 = -\frac{K}{r} - \partial_t \lambda \tag{11}$$

which is impossible as $K > 0$, and $\partial_t \lambda$ must be sometimes positive for any periodic solution. Hence, by contradiction, there does not exist a steady state.

Addendum While the above solution is mathematically rigorous, we received a submission from the team "rotting crustaceans" which in our opinion is also rigorous while being more intuitive. We attach a modified sketch of their solution below.

For the viscous fluid the viscous dissipation function is defined as $\Phi = \tau_{ij}\partial_i u_j$, where Einstein notation is used and τ_{ij} is the stress tensor

$$\tau_{ij} = \mu(\partial_i u_j + \partial_j u_i) + \lambda(\nabla \cdot \mathbf{u})\delta_{ij} \tag{12}$$

At any sort of energy minimum we must have Φ zero everywhere, which implies rigid body motion. Moreover, for a given fixed density distribution with fixed total angular momentum, the minimum energy configuration is rigid body. This can be proved by considering the KE:

$$T = \iiint_V \frac{1}{2}\rho \mathbf{u}^2 dV = \iiint_V \frac{1}{2}\mathbf{u}^2 dm \tag{13}$$

while the angular momentum is

$$L = \iiint_V r u_\phi dm \tag{14}$$

where we have defined $\mathbf{L}$ to be along the z-axis. Then we have

$$\iiint_V I u_\phi^2 dm = \iiint_V u_\phi dm \iiint_V r^2 dm \geq (\iiint_V r u_\phi dm)^2 = L^2 \tag{15}$$

by Cauchy-Schwarz inequality, and therefore

$$T \geq \iiint_V \frac{1}{2} u_\phi^2 dm \geq \frac{1}{2}\frac{L^2}{I} \tag{16}$$

completing the proof. This therefore implies that if there existed an energy minimum which is not rigid body, one could keep the density distribution the same and instead rotate the body rigidly. (This also eliminates any time-varying solutions.)

Now, for the rigid body solutions, we can show that they cannot be an energy minimum by considering the fact that, for any given particle orbiting around a central mass of mass M at a distance r and with angular momentum l, it has energy

$$\epsilon = \frac{l^2}{2r} - \frac{GM}{r} \tag{17}$$

This is negative by the boundedness of the mass, and therefore if the mass is moved inwards (decreasing r) while conserving angular momentum, the energy will decrease. We can therefore imagine a situation where, if there were a given rigidly rotating density distribution which is a local energy minimum, one can move a parcel of mass inwards while conserving the angular momentum to further decrease the energy. We can then redistribute the angular momentum so that the motion is again rigid body - as there is a slight decrease in moment of inertia, ω will increase slightly and the angular momentum is transferred outwards - and so on and so forth, until all the mass is concentrated at the central mass and an infinitesimal mass is orbiting infinitely far away and/or at infinite angular velocity. This is clearly unphysical and therefore there does not exist any configuration which is an energy minimum and stable.

Theory Question 3 [10 marks]

On Ice [10 marks]

A skater is skating on ice on a dry day with ambient temperature T_0. The skater has mass $M = 60$ kg and the skates have total area $A_{skate} = 5 \times 10^{-4}$ m^2 and length $L = 10$ cm in contact with the ice. The skater is moving forward at a velocity $v = 1\frac{\text{m}}{\text{s}}$. You may assume throughout the question that the stakes are perfectly vertical.

The goal of this question is to consider the physical effects that affect skating.

(A.1) Estimate the melting point of the ice directly under the skate in terms of the given quantities, the density of water at $0°C$ $\rho_w = 0.9998\frac{\text{g}}{\text{cm}^3}$, the density of ice at $0°C$ $\rho_i = 0.9168\frac{\text{g}}{\text{cm}^3}$, the latent heat of fusion for water $L_f = 334\frac{\text{kJ}}{\text{kg}}$, and $T_m = 273.15$ K is the temperature at which ice melts under standard conditions. [0.8 pts]

(A.2) In what regime do you expect the melting of the ice due to the pressure of the blade to be significant? [0.2 pts]

For part (B), we assume that T_0 lies within the regime where the effect we considered in (A) is significant. Do **not** make this assumption for the remainder of the question.

Aside from the pressure of the blade, intermolecular forces also play a role to create meltwater at the ice surface. The force between a molecule of water vapour and a molecule of ice with a liquid layer separating them can be modelled by

$$U(r) = \frac{C}{r^6} \tag{18}$$

where C is a constant determining the strength of the intermolecular forces, and r is the separation between the two molecules.

(B.1) Considering only the interaction between the vapour and ice layers, find the potential energy per unit area of a layer of liquid layer of height h, defining any quantities you need and stating any simplifying assumptions made. [1 pt]

(B.2) Estimate the equilibrium thickness of the liquid layer h due to intermolecular forces. You are given the strength of interaction $C = 5.35 \times 10^{-74}$ J m^6, the molar mass of water $\mu = 18\frac{\text{g}}{\text{mol}}$, the latent heat of vaporisation of water $L_v = 2260\frac{\text{kJ}}{\text{kg}}$, and atmospheric pressure $P_0 = 1 \times 10^5$ Pa. [1.4 pts]

(B.3) Estimate the thickness of the liquid layer caused by pressure melting. The heat capacity of ice is $c = 2.09\frac{\text{kJ}}{\text{kg K}}$ and the thermal conductivity of ice is $\kappa = 2.18\frac{\text{W}}{\text{m K}}$. When does each effect dominate? [0.8 pts]

(B.4) Based on these answers, estimate the frictional force experienced by the skater near $0°C$. The viscosity of water is $\nu = 1.79 \times 10^{-3}\frac{\text{kg}}{\text{m s}}$. What does this tell you about the effects governing the frictional force experienced by the skater? [0.4 pts]

We now consider the correction caused by the frictional heat generated while the blade is sliding. This heat is conducted into the ice, which melts some of it. In previous sections we have assumed that the thickness of the liquid layer is constant across the length of the blade, but this is not the case once friction is taken into account. In this part, you may assume that the blade glides on top of the liquid layer.

(C.1) By considering the power dissipated by friction in shear flow in the liquid layer, write down an expression for the rate of change of the thickness along the blade length, i.e. as the ice spends more time in contact with the blade. [0.4 pts]

(C.2) Assume that as the blade comes into contact with the ice, an infinitesimal component of the ice is immediately molten and heated to $0°C$. By considering the vertical heat transfer into the ice block, find the rate of change of thickness along the blade length due to heat conduction into the ice. [1.4 pts]

(C.3) Combining the two effects, find an expression relating the thickness of the liquid layer $h(x)$ with the distance along the blade x. How does this compare to the effect considered in part B? [0.8 pts]

(C.4) Find an expression for the total frictional force experienced by the blade. How does this scale at low and high temperatures? [0.6 pts]

(C.5) Using this model, explain why we cannot skate on wax at room temperature. [0.4 pts]

In previous sections we have considered that the blade remains on top of the liquid layer. However, this is not in reality the case. In this section we consider the corrections caused by the effect of the blade sinking into the liquid layer.

(D.1) Consider a stationary blade which is sinking into a water layer of height h at a rate of $\dot{h}$. Neglecting the vertical pressure variation, any obstacles to the side of the blade, and the fluid movement along the blade, find $p(y, z)$, the pressure field as a function of the vertical coordinate and the lateral coordinate. [1 pt]

(D.2) Find the condition for the lateral flow to have a small effect on the height profile. Find the minimum velocity for this criteria to be satisfied. [0.8 pts]

On Ice - Solution

(A.1) We wish to find the new melting point of the ice under the skate's pressure. This can be found via the Clausius-Clayperon relation:

$$\frac{dP}{dT} = -\frac{L_f}{T\Delta v} \tag{19}$$

(0.2) which gives the new melting point in the linear approximation as

$$P - P_0 \approx P = -(T^* - T_m)\frac{L_f}{T_m \Delta v} \tag{20}$$

$$T^* = T_m(1 - \frac{P\Delta v}{L_f}) \tag{21}$$

(0.2) where Δv is the change in specific volume and $P = \frac{Mg}{A_{skate}}$ is the pressure applied on the ice. Substituting in numerical values gives $P = 1.18 \times 10^6$ Pa, $T^* = -0.088°C$. (0.2)

(A.2) Only at temperatures very close to T_m will the pressure melting effect be significant. (0.2)

(B.1) The total potential energy of the two fluid layers due to intermolecular forces is expressed through integrating over both layers:

$$U = C\rho_{vap}\rho_{ice} \iiint_{V_{vap}} \iiint_{V_{ice}} \frac{1}{|\mathbf{r} - \mathbf{r}'|^6} d^3\mathbf{r} d^3\mathbf{r}' \tag{22}$$

(0.2) Where ρ_{vap}, ρ_{ice} are the number densities of the respective phases. Decomposing into Cartesian coordinates, fixing the ice atom at x=0, y=0 without loss of generality,

$$U = C\rho_{vap}\rho_{ice} \int_{-\infty}^{0} dz_1 \int_h^{\infty} dz_2 \iint dxdy \frac{1}{(x^2 + y^2 + (z_1 - z_2)^2)^3} \tag{23}$$

$$U = C\rho_{vap}\rho_{ice} \int_{-\infty}^{0} dz_1 \int_h^{\infty} dz_2 \frac{\pi}{2} \frac{1}{(z_1 - z_2)^4} \tag{24}$$

$$U = \frac{\pi}{12h^2} C\rho_{vap}\rho_{ice} \tag{25}$$

(0.4), 0.2 for integrating over the plane This assumes that both the gas and ice phase have approximately constant number densities. (0.2)

(B.2) Thermodynamic equilibrium occurs when the Gibbs free energy is minimised. (0.2) At the phase transition we have

$$(\frac{\partial G}{\partial T})_P = -S = \frac{L_f}{T_m} \tag{26}$$

so the bulk free energy of the liquid layer per unit area is

$$\Delta G = \frac{L_f \Delta T}{T_m} h \tag{27}$$

(0.2) where ΔT is the difference in ambient temperature from the melting point. The change in enthalpy is equal to the potential energy caused by the intermolecular forces. As a result the full expression for the Gibbs energy is

$$\Delta G = h(\frac{L_f \Delta T}{T_m} + \frac{\pi C\rho_{vap}\rho_{ice}}{12h^3}) \tag{28}$$

Minimising this with respect to h, we get

$$\frac{d\Delta G}{dh} = \frac{L_f \Delta T}{T_m} - \frac{\pi C\rho_{vap}\rho_{ice}}{6h^3} = 0 \tag{29}$$

$$h = (\frac{\pi C\rho_{vap}\rho_{ice} T_m}{6L_f \Delta T})^{\frac{1}{3}} \tag{30}$$

(0.2) Next, we need to find ρ_{vap} and ρ_{ice}. The mass density of ice at $0°C$ was given as $0.9168\frac{\text{g}}{\text{cm}^3}$, so using the molar mass of water $\mu = 18\frac{\text{g}}{\text{mol}}$ we get $\rho_{ice} = 3.07 \times 10^{28}$ m^{-3}. (0.2) For water vapour, we need the vapour pressure of water at $0°C$. Again we use Clausius-Clayperon but this time with the latent heat of vaporisation, and noting that the specific volume of the gas is much larger than that of the liquid:

$$\frac{dP}{dT} = \frac{L_v P \mu}{RT^2} \tag{31}$$

$$P = P_0 \exp\left(-\frac{L_v \mu}{R}(\frac{1}{T} - \frac{1}{T_0})\right) \tag{32}$$

where P_0 is atmospheric pressure and $T_0 = 373$ K is the boiling point of water. Substituting in the appropriate values we get at $0°C$ that $P = 819$ Pa. (Note: L varies significantly over the range of 0 to $100°C$, so this calculation is only approximate; the true value for vapour pressure at $0°C$ is 611 Pa. Calculations using this value should not be penalised.) (0.2)

The number density of water vapour is therefore $\rho_{vap} = \frac{P_v}{k_B T} = 2.16 \times 10^{23}$ m^{-3}. (True value: 1.62×10^{23}.) Substituting the values, we get (0.1)

$$h = (\frac{1.518 \times 10^{-25}}{\Delta T})^{\frac{1}{3}} \tag{33}$$

For the maximum value of $\Delta T = 0.088$, we get $h = 12.0$ nm. (Correct value: $h = 10.9$ nm.) (0.1)

(B.3) There are multiple valid ways to do this, and all answers within the same order of magnitude should be accepted. We consider the amount of heat transmitted by a medium with a temperature difference $\Delta T_m = 0.088$ K in the time the blade presses on the ice. The thermal diffusion lengthscale is $l_{diff} = \sqrt{\frac{DL}{v}} = \sqrt{\frac{\kappa L}{\rho c v}} = 3.37 \times 10^{-4}$ m. (0.2) This is the approximate depth to which the ice will be cooled. To find the heat transmitted per unit area, we use the Fourier law: $\dot{Q} = \kappa \frac{\Delta T}{l_{diff}} = 569\frac{\text{W}}{\text{m}^2}$. (0.2) This can melt a thickness of $h = \frac{\dot{Q}t}{\rho L_f} = 186$ nm of ice, corresponding to 170 nm of water due to increased density. (0.1) At these very high temperatures (very close to $0°C$), pressure melting is more significant. However pressure melting has $h \propto T - T_{m,p}$ where $T_{m,p}$ is the melting point of the ice under the pressure, whereas the liquid layer formed by intermolecular forces has $h \propto \frac{1}{(T-T_m)^{-\frac{1}{3}}}$. This means that as $T \to T_{m,p}$ the intermolecular force will start to dominate, and below $0.088°C$ pressure melting does not work at all. (0.2)

(B.4) We take the larger value of $h = 170$ nm. The frictional force originates from the shear viscosity of the water. The total force is $f = \nu A \frac{v}{h} = 5.25$ N. (0.2) This is not a lot, but we know that at $0°C$ ice is extremely slippery, so a frictional force corresponding to approximately 1% of the body weight while gliding at a very slow speed is still too high. There must be other effects which thicken the water layer and lower the friction. (0.2) In fact, the intelligent reader will realise that the frictional heat generated per unit area is $9.6\frac{\text{kW}}{\text{m}^2}$, which is quite a lot higher than the $569\frac{\text{W}}{\text{m}^2}$ involved in heat transfers during pressure melting. This gives a hint that frictional heating is the dominant mechanism in skating, which will be explored in the next section.

(C.1) (Part C of this question is based on Lozowski and Szilder 2009, Lozowski and Szilder 2013.) Let x be the distance along the blade under consideration. In a unit time the power dissipated at this position is

$$dP = \frac{\nu v^2}{h(x)} w dx \tag{34}$$

where $w = \frac{A_{skate}}{2L}$ is the width of the skate. (0.2) The power required to melt a layer of water of depth dh is

$$dP = \rho_w v w L_f dh \tag{35}$$

equating these quantities gives (0.2)

$$\frac{dh}{dx} = \frac{\nu v}{h \rho_w L_f} \tag{36}$$

(C.2) We first need to consider the thermal model given: as the blade comes into contact with the ice at time $t = 0$, the boundary layer temperature is instantaneously raised to $0°C$. The thermal diffusion equation gives:

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial y^2} \tag{37}$$

(0.2) In the vertical profile, this admits a similarity solution $T = f(z = \frac{y}{\sqrt{Dt}})$. (0.2) f satisfies the differential equation

$$-\frac{1}{2}\frac{y}{\sqrt{D}t^{\frac{3}{2}}}\frac{df}{dz} = D\frac{1}{Dt}\frac{d^2 f}{dz^2} \tag{38}$$

$$-\frac{1}{2}z\frac{df}{dz} = \frac{d^2 f}{dz^2} \tag{39}$$

The solution to this equation is (0.1)

$$\frac{df}{dz} = Ce^{-\frac{1}{4}z^2} \tag{40}$$

(0.1) or, in other words,

$$\frac{dT}{dy} = \frac{C}{\sqrt{Dt}}e^{-\frac{1}{4}\frac{y^2}{Dt}} \tag{41}$$

To find the constant C, we note that the boundary condition gives $T(y=0) = T_m$ and $T(y \to \infty) = T_0$. We note that

$$T(y \to \infty) - T(y=0) = \frac{C}{\sqrt{Dt}}\int_0^\infty e^{-\frac{y^2}{4Dt}}dy = \frac{C}{\sqrt{Dt}}\sqrt{\pi Dt} = \sqrt{\pi}C \tag{42}$$

(0.2) so $C = \frac{1}{\sqrt{2\pi}}(T_m - T_0)$. This means that the instantaneous heat flux at the surface into the ice per unit area is

$$\dot{Q} = \kappa\frac{T_m - T_0}{\sqrt{\pi Dt}} \tag{43}$$

where $t = \frac{x}{v}$ is the time the ice has spent in contact with the blade. (0.2) Equating this with (18) we obtain (0.1)

$$\frac{dh}{dx} = -\frac{\kappa(T_m - T_0)}{\rho_w L_f\sqrt{\pi Dxv}} \tag{44}$$

(C.3) The combined differential equation is (0.1)

$$\frac{dh}{dx} = \frac{\nu v}{h\rho_w L_f} - \frac{\kappa(T_m - T_0)}{\rho_w L_f\sqrt{\pi Dxv}} \tag{45}$$

We let the constants $C_1 = \frac{\nu v}{\rho_w L_f}$ and $C_2 = \frac{\kappa(T_m - T_0)}{\rho_w L_f\sqrt{\pi Dv}}$ so that

$$\frac{dh}{dx} - \frac{C_1}{h} = -\frac{C_2}{\sqrt{x}} \tag{46}$$

We note that, substituting $h = K_2 x^{\frac{1}{2}}$, we get

$$\frac{K_2}{2}x^{-\frac{1}{2}} - \frac{C_1}{K_2}x^{-\frac{1}{2}} = -C_2 x^{-\frac{1}{2}} \tag{47}$$

so $K_2 = -C_2 \pm \sqrt{C_2^2 + 2C_1}$. (0.3) With the initial condition $h(0) = 0$ we get

$$h = (-C_2 + \sqrt{C_2^2 + 2C_1})\sqrt{x} \tag{48}$$

where we choose the positive sign to ensure the coefficient is always larger than 0. (0.1) If we substitute $\Delta T = 0.088$, we obtain $C_1 = 5.36 \times 10^{-12}$ and $C_2 = 3.03 \times 10^{-7}$, so the coefficient is 2.98×10^{-6}. (0.1) The deepest part of the liquid layer will have depth 0.94 μm. This is much larger than the nanoscale effects in part B. (0.1)

(C.4) The total frictional force is expressed by the integral (0.2)

$$f = \nu wv\int_0^L \frac{1}{h}dx = \nu wv(-C_2 + \sqrt{C_2^2 + 2C_1})^{-1}(2\sqrt{L}) \tag{49}$$

At very high temperatures (i.e. very close to $0°C$), we have $C_2 = 0$ so $f = 2\nu w v\sqrt{\frac{L}{2C_1}}(\approx 1.728\text{ N})$. (0.1) At very low temperatures, $C_2 \to \infty$ (0.1) so to lowest order in $\frac{C_1}{C_2}$ we have:

$$f = 2\nu w v\frac{C_2}{C_1}\sqrt{L} \tag{50}$$

so the force is proportional to $T_m - T_0$. (0.2)

(C.5) Note that $\frac{C_1}{C_2} = \frac{\nu v\sqrt{\pi D v}}{\kappa(T_m - T_0)}$. Wax has a higher viscosity and higher thermal diffusivity, as well as lower thermal conductivity. The liquid layer formed is too thick, the blade sinks into it and gets stuck. (Other physical answers are also accepted, including explanations that the blade pressure inhibits melting, wax is a soft material which plastically deforms, etc.) (0.4)

(D.1) The governing equations of the fluid flow are

$$\frac{\partial p}{\partial y} = \nu\frac{\partial^2 v_y}{\partial z^2} \tag{51}$$

$$\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{52}$$

(these can be derived from differential element analysis if needed, but it is acceptable to quote them as standard equations of motion.) (0.2) We approximate that the vertical variation of pressure is negligible so that v_y can be written as

$$v_y(y,z) = \frac{1}{2\nu}\frac{\partial p}{\partial y}(z^2 - zh) \tag{53}$$

(0.2) substituting in the boundary conditions $v_y(0) = v_y(h) = 0$. Substituting this expression in the continuity equation, we obtain

$$\frac{\partial^2 p}{\partial y^2} = \frac{12\nu\dot{h}}{h^3} \tag{54}$$

(0.2) where we have also used the fact that $\frac{\partial^2 v_z}{\partial z^2} = 0$, and integrated the expression over z from 0 to h. This gives (0.2)

$$p = \frac{6\nu\dot{h}}{h^3}(\frac{w^2}{4} - y^2) \tag{55}$$

(D.2) The upward force experienced per unit length by the blade due to the pressure from the excess flow is the expression in (D.1) integrated over the width of the blade: (0.2)

$$F = \frac{6\nu\dot{h}}{h^3}2\int_0^{\frac{w}{2}}(\frac{w^2}{4} - y^2)dy = \frac{\nu\dot{h}w^3}{h^3} \tag{56}$$

In other words, the thinning of the liquid layer $\dot{h} = -v\frac{dh}{dx}$ is determined by sufficient pressure to support the weight of the skater, so that

$$\frac{dh}{dx} = -\frac{Ph^3}{\nu w^2 v} \tag{57}$$

with P as defined in (A.1). (0.2) For this to be smaller than the other effects, we need $\frac{Ph^3}{\nu w^2 v} < \frac{h}{2x}$ or

$$v > \frac{2P(-C_2 + \sqrt{C_2^2 + 2C_1})^2 L^2}{\nu w^2} \tag{58}$$

where we have replaced x with L to represent the maximum value. (0.2) For the given numerical values, this is $2.33\frac{\text{m}}{\text{s}}$.