2023 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 3 long answer questions and 160 points over 2 full days from August 5, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form.**
- If you wish to request a clarification, please use this form. To see all clarifications, view this document.
- Participants are given a google form where they are allowed to submit up-to 100 megabytes of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.
- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.
- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
Universal gas constant	$R = 8.31$ J/(mol $\cdot$ K)
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	1 eV $= 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}$ (N $\cdot$ m^2)/kg^2
Acceleration due to gravity	$g = 9.81$ m/s^2
1 unified atomic mass unit	1 u $= 1.66 \cdot 10^{-27}$ kg $= 931$ MeV/c^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J $\cdot$ s $= 4.41 \cdot 10^{-15}$ eV $\cdot$ s
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(N $\cdot$ m^2)
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}$ (N $\cdot$ m^2)/C^2
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}$ W/m^2/K^4

T1: Booster

In this problem, we explore a simplified model of Solid Rocket Boosters (SRBs). SRBs are supplements to liquid rockets and provide enormous thrust at liftoff at the expense of lower specific impulse. They have a pretty simple design – a tube containing solid propellant with a central cavity that acts as a combustion chamber. As the solid fuel burns (deflagrates), gasses are forced out of the combustion chamber through a nozzle, producing thrust. It is desirable to choose a combustion chamber design which produces constant thrust (to reduce structural load on the spacecraft) and also have constant internal pressure and temperature (to reduce stress on the SRB). Below a diagram, along with a typical shape of the combustion chamber.

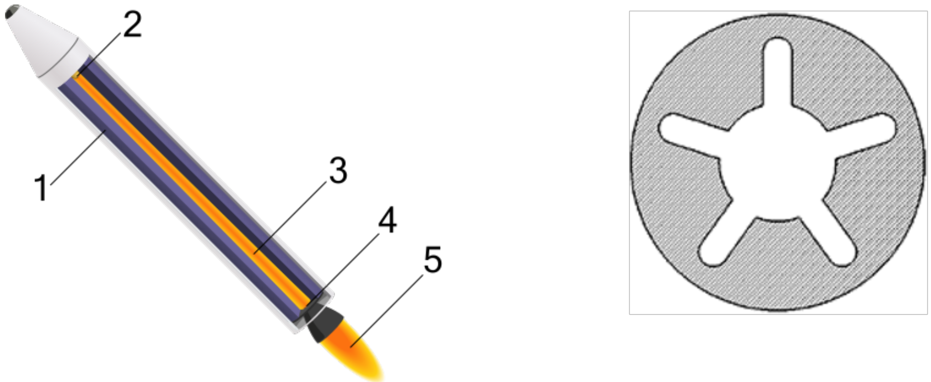
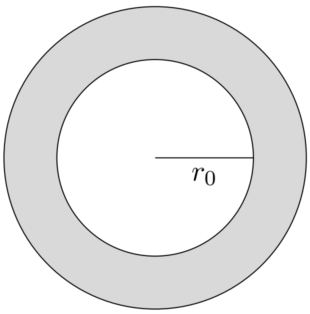


Figure 1: The structure of an SRB (left) and the shape of a typical chamber (right)

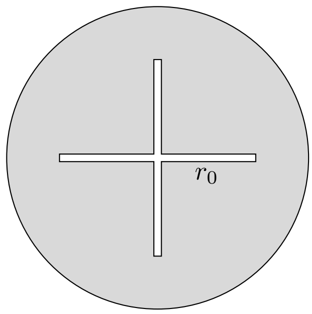
Data:
Rate of vaporization of surface of fuel: $v = 9 \times 10^{-3} \frac{\text{m}}{\text{s}}$
Length of combustion chamber: $l = 45\text{m}$
Radius of combustion chamber: $r_0 = 0.6\text{m}$
Density of propellant: $\rho = 1500 \frac{\text{kg}}{\text{m}^3}$
Molar mass of exhaust gas: $M = 0.040 \frac{\text{kg}}{\text{mol}}$
Temperature inside chamber: $T = 4000$ K
Ideal gas constant: $R = 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}}$

(a) Find the burn rate of fuel (in terms of $\frac{\text{kg}}{\text{s}}$) as a function of time for the following designs for the combustion chamber. Assume that the combustion chamber never reaches the walls of the SRB.

(i)



(ii)



Answer the following for design i.

(b) Compute the thrust as a function of time. Assume the temperature and pressure of the chamber stay constant, and the density of the gas is ρ_g.

(c) Is the assumption of constant internal conditions for this design valid?

(d) Now assume that the temperature is constant, but the pressure changes. Estimate to within a factor of 2 the pressure inside the chamber as a function of time.

T2: The Complex Potential

A Conformal Transformation

A complex number can be thought of as a mathematical object that "stores" two real numbers. An element z of the set of complex numbers $\mathbb{C}$ can be written as:

$$z = x + iy$$

where $i \equiv \sqrt{-1}$ is called the imaginary unit, and $x, y \in \mathbb{R}$. With this, one can think of constructing complex-valued functions – namely, a function $f : \mathbb{C} \to \mathbb{C}$:

$$f(z) = f(x + iy) = w(x, y) + iu(x, y) \quad (1) \tag{1}$$

Since complex numbers encode two real numbers x and y, we can consider complex-valued functions yielding two real-valued functions $w(x, y)$ and $u(x, y)$ that depend on x, y. w and u, being real-valued multivariable functions, allow us to extend calculus on $\mathbb{R}^2$ to $\mathbb{C}$. You are given that the composition of two complex differentiable functions yields another complex differentiable function on the appropriate domain.

(a) Starting from the definition of a derivative in single-variable calculus, show that if f given in (1) is complex differentiable, w and u satisfy:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial w}{\partial y} = -\frac{\partial u}{\partial x}$$

(b) Show that $w(x, y)$ and $u(x, y)$ are solutions to the 2D Laplace equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2}$$

(c) A parametrized curve γ on the complex plane can be thought of as a function that takes a real number t and yields a point $\gamma(t)$ on the complex plane. Two parametrized curves γ_1 and γ_2 intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$; tangent lines of the two curves at p form an angle α. Show that the tangent lines of the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(p)$ form the same angle α, where $\circ$ is the composition function.

A Conformal Transformed Electrostatic System

We know from electrostatics that in a charge-free region, the electrostatic potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$. As you've shown above, this motivates the idea of defining a complex differentiable function whose real part is the electrostatic potential – complex differentiability encodes the fact that the electrostatic potential satisfies (2). Given a 2D electrostatics problem in a charge-free region, we define the *complex potential* $f(z)$ as:

$$f(z) = \phi(x, y) + i\psi(x, y)$$

where ψ is an appropriate real-valued function that satisfies the conditions derived in question 1, making f a complex differentiable function. You are given the uniqueness theorem: the 2D Laplace equation has a unique solution that satisfies specified boundary conditions.

(d) Given a 2D electrostatics problem and an appropriate complex potential f, write the derivative $\frac{\mathrm{d}f}{\mathrm{d}z}$ in terms of the x and y components of the electric field: $\mathbf{E}(x, y) = E_x(x, y)\hat{x} + E_y(x, y)\hat{y}$.

(e) Show that the function $\phi(x, y) = \frac{\phi_0}{\pi} \arctan \frac{y}{x}$ satisfies the 2D Laplace equation with the boundary condition $\phi(x > 0, 0) = 0$ while $\phi(x < 0, 0) = \phi_0$.

(f) The complex logarithm function is defined at $z = re^{i\theta}$ as $\log(z) = \log |r| + i\theta$ for $r \neq 0$, $0 < \theta < 2\pi$. Show that the upper-half plane $\mathcal{H}$ (points on the complex plane with a positive imaginary part), under the logarithm map, turns into a strip bounded by $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \pi$. You may assume that the logarithm function is complex differentiable on $\mathcal{H}$ and that its inverse is e^z for this problem.

(g) Starting from the electrostatic potential in (e), find the electrostatic potential in the region between two infinitely large capacitor plates separated by distance d. Carefully argue why this method yields the correct solution.

(h) Show that the function $f(z) = \frac{i(1-z)}{(1+z)}$ maps the unit circle $\mathcal{C}$, centered at the origin, to the real line and the interior of the circle to $\mathcal{H}$. Determine the inverse g of the function f and explain why there's a one-to-one correspondence between points bounded by $\mathcal{C}$ and $\mathcal{H}$.

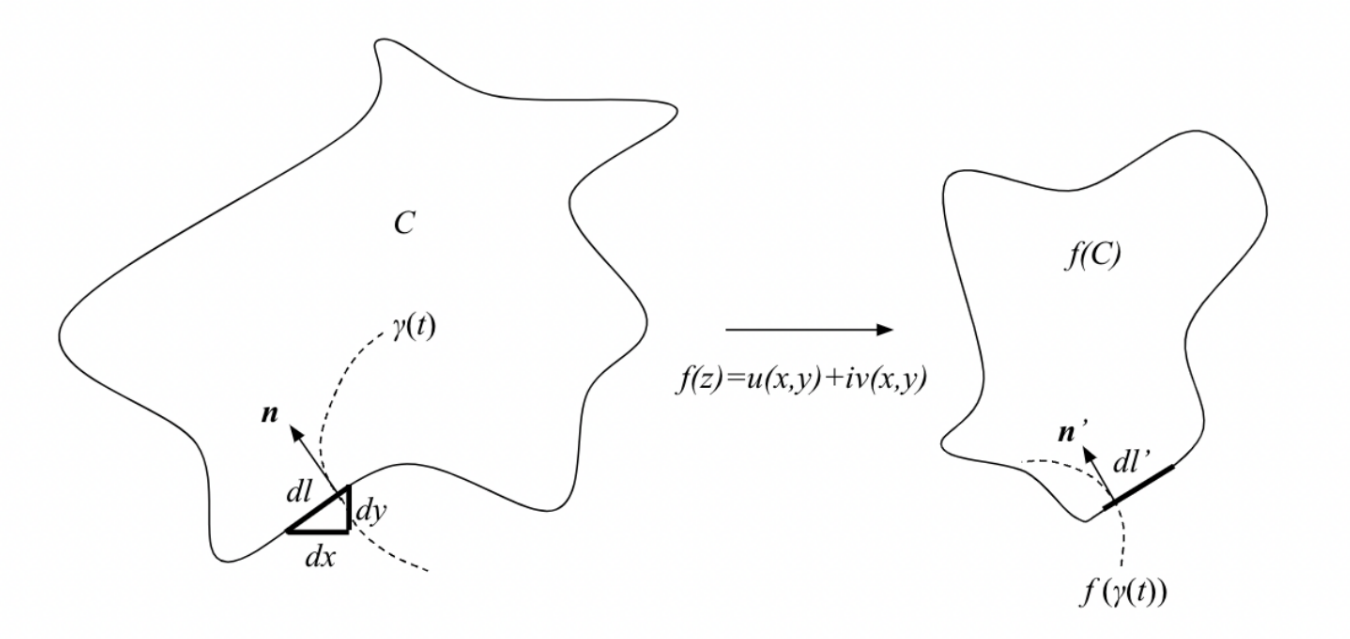
(i) Find the electrostatic potential in the region between an infinite rod of charge density λ and a grounded concentric conducting cylindrical shell of radius R.

(j) Now consider the same infinite rod suspended above a grounded conducting plate at a height h. Use your results for questions (h) and (i) to derive the electrostatic potential ϕ in the region above the plate.

(k) Consider the curve C shown in the figure below. You are given that the image of $\mathrm{d}l$ is $\mathrm{d}l'$. If we take some line element $\mathrm{d}l$ on the curve C, find the scaling factor to its image $\mathrm{d}l'$ on $f(C)$. This is known as a conformal transformation in which the mapping is a holomorphic function.

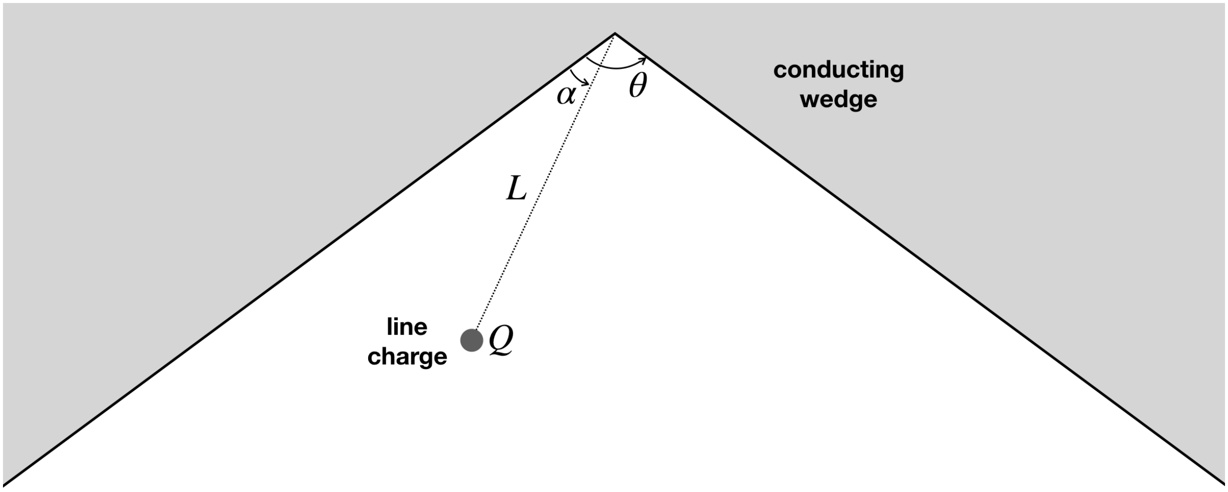
(l) Take C to represent a conductive surface. If ϕ is an appropriate solution to the 2D Laplace equation satisfying $\phi|_C = 0$, find expressions for the surface at $\mathrm{d}l$, the electrostatic potential ϕ' satisfying $\phi'|_{f(C)} = 0$, and the surface charge at $\mathrm{d}l'$ from ϕ'.

(m) Verify that your results for questions (k) and (l) are consistent.



On the Electrostatic Interaction between a Line Charge and a Conducting Wedge

We can apply the knowledge we have gained here to solve a familiar physical puzzle. This puzzle has been widely known and popular in specific cases (e.g. easily solvable using the image method), but it lacks generality. Consider a grounded, infinitely large, conducting wedge with an opening angle θ. Inside the wedge, there is a thin, infinitely long, straight line with a uniform charge density of Q per unit length, positioned parallel to the edge of the wedge. The distance between the line and the edge is L, the plane that pass through both the line and the edge make an angle α with a face of the wedge.



(n) Find the direction and the magnitude of the electrical force acting on the line per unit length. Solve for the general case of $2\pi > \theta > \alpha$, then evaluate your answer in the unit of kQ^2/L, for $\theta = 120^o$ and $\alpha = 30^o$.

T3: General Relativity

The Alphabet of Spacetime Metrics

In multivariable calculus, we study the generalization of line elements to various coordinate systems. A coordinate system in $\mathbb{R}^3$ can be described by a function ψ that takes a point $(x, y, z) \in \mathbb{R}^3$ and outputs the three coordinates describing the same point in the coordinate system. For example, the ψ describing spherical coordinates would be defined as $\psi(x, y, z) = (r, \theta, \phi)$. As there must be a unique representation of each point in $\mathbb{R}^3$ in the ψ coordinates (ex. $r = 3, \theta = \frac{\pi}{2}, \phi = \pi$ cannot correspond to two points on the 3D space) there must be a one-to-one correspondence between points in $\mathbb{R}^3$ and points described in ψ coordinates. Hence, there's a well defined inverse function ψ^{-1}. In spherical coordinates, for instance, $\psi^{-1}(r, \theta, \phi) = (x, y, z)$.

Suppose we have a coordinate system $\psi(x, y, z) = (q_1, q_2, q_3)$. We ask a very important question: if we make an increment of dq_i in each q_i in the ψ coordinates, what's the physical distance between the starting and the finishing points in $\mathbb{R}^3$? In other words, what's the distance between the points $\psi^{-1}(q_1 + dq_1, q_2 + dq_2, q_3 + dq_3)$ and $\psi^{-1}(q_1, q_2, q_3)$? This distance is known as the *line element* generated by a coordinate system ψ. If the coordinate system is *orthogonal*, meaning the basis vectors $\hat{q_1}$, $\hat{q_2}$, $\hat{q_3}$ are all mutually perpendicular, we know that the physical displacements in taking $q_i \to q_i + dq_i$ are in perpendicular directions in $\mathbb{R}^3$, so we can find the line element by Pythagoras's theorem.

(a) Show that for an orthogonal coordinate system ψ in an arbitrary $\mathbb{R}^n$, the square of the line element ds^2 has the form $\sum_{i=1}^{n} g_i(q_1, ..., q_n)dq_i^2$ where each g_i is a function of $q_1, ..., q_n$. Find an expression for g_i in terms of the derivatives of ψ or ψ^{-1}.

We can think of physical observers as coordinate systems. If an arbitrary observer moving in space-time measures an event (t, x, y, z), the coordinates seen by the observer can be expressed as $\psi(t, x, y, z)$ by some function ψ. It turns out that special relativity can be described with the math we've developed so far. In particular, we can consider measuring space-time lengths by integrating the line element along paths in space-time. You are given that for *any* observer with the coordinate system $\psi \equiv (t, x, y, z)$, the line element is given by $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$

(b) It's a well-known result of special relativity that space-time lengths are measured to be the same regardless of the frame. Resolve the twin paradox by integrating the line element along the spaceship twin's world-line.

It turns out that physics in weak gravitational fields can also be described with the math we've developed so far. Consider a 1 dimensional system with a time-independent gravitational potential $\Phi(x)$ everywhere.

(c) An important postulate of general relativity states that observers that move solely under the influence of gravity take the path with the longest space-time length. Suppose an observer of this system takes a path parameterized by a variable $\sigma \in [0, \sigma_f]$: $(t(\sigma), x(\sigma), y(\sigma), z(\sigma))$. Show that the line element $ds^2 = -(1+2\Phi(x))dt^2+(1-2\Phi(x))dx^2$ generates equations of motion that are consistent with Newton's laws (neglecting special relativity).

Hint: Refer to the Euler-Lagrange equation, but you do NOT have to perform the derivation to solve this problem.

(d) Derive a statement of time dilation at a point (x, y, z).

The Spacetime Metric of a Black Hole

As we saw in part 1, the Minkowski metric describes geometry of spacetime in special relativity and is given as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

In general relativity, the result is more complex, and we describe spacetime via the Schwarzchild metric, a solution of Einstein's equations under spherical symmetry. The spacetime interval in spherical coordinates can be expressed in terms of proper time τ and the speed of light c as

$$ds^2 = -c^2d\tau^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where t is the coordinate time, r is the radial coordinate, θ is the polar angle, ϕ is the azithumal coordinate, and r_s is the Schwarzchild radius of a massive body. The Schwarzchild radius is related to the body's mass M by $r_s = \frac{2GM}{c^2}$ where G is the universal gravitational constant.

General relativity describes gravity as the consequence of curving spacetime. Geodesics generalize straight lines to curved spacetime and are very useful as a result. In the following problem, the radial geodesic equation may be helpful:

$$\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \Gamma^r_{\mu\nu}\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = 0$$

where Γ represents a Christoffel symbol. The relevant terms of Γ are Γ^r_{tt} , Γ^r_{rr} , $\Gamma^r_{\theta\theta}$, and $\Gamma^r_{\phi\phi}$. Define $B = 1 - \frac{r_s}{r}$, then we can get:

$$\Gamma^r_{tt} = \frac{c^2B\frac{\mathrm{d}B}{\mathrm{d}r}}{2}$$
$$\Gamma^r_{rr} = -\frac{B^{-1}\frac{\mathrm{d}B}{\mathrm{d}r}}{2}$$
$$\Gamma^r_{\theta\theta} = -rB$$
$$\Gamma^r_{\phi\phi} = -Br\sin^2\theta$$

The Photon Sphere

In 2019, the first image of a black hole was taken in 2019 by the Event Horizon Telescope (EHT) at the center of the galaxy M87. The historic image of the black hole showcased a glowing ring of light surrounding the dark abyss known as the "photon ring." In this region, light rays follow closed circular paths due to the strong gravitational pull of the black hole. When a photon follows a closed circular path on the photon ring, it can effectively orbit the black hole multiple times before either escaping to infinity or being captured by the event horizon. This phenomenon is known as the "photon sphere."

Figure 2: Event Horizon Telescope Collaboration

You may recall the massive outburst of media to the images of M87. Many were already talking about it before the images came out, like Veritasium. Take 10 minutes to watch the video.

In this problem, we will be analyzing the optical effects for an observer next to a black hole. Assume for an idealized case that the black hole has a mass $M = 15M_\odot$ and is uncharged and not rotating.

(e) Prove that the minimum radius of the black hole's photon ring follows the relationship $r_p = \frac{3}{2}r_s$.

(f) Bill is near the black hole a distance r_p away. Assume Bill's width is $w = 2\delta$ where $\delta = 0.25$ m. Sketch light ray trajectories showing how Bill could see his own back. Determine the range for the angle of view α enabling this.

(g) Approximately what is the orbital period (in miliseconds) of a singular photon in coordinate time t around the black hole. At what mass of a black hole would this time be greater than the visual reaction time of a human $t_r = 150$ ms (meaning if you closed your eyes, it would take some time for you to visualize your back again)? Is this mass possible?

(h) Propose a method (realistic or not) for Bill to safely get close to the black hole without getting ripped apart by gravity. If you believe there is no possible method, then give your reasoning behind why. You might want to consult various sources for information. To simplify the process and avoid teams searching for data such as "shear stress of a human", assume that Bill is a cylindrical object made of a material of your choosing that has a uniform density, a radius δ, and a length of $L = 1.75$ m. Bill possesses exceptional engineering skills, enabling him to construct virtually anything, provided it doesn't contravene the principles of physics.

(i) Let's assume that the galaxy has N_{star} stars, each emitting light isotropically with the same total power L. Assume the stars are distributed uniformly throughout a spherical volume of radius R. For an ideal case where the path of the light from the stars is not obstructed by any other celestial objects, estimate the expected number of photons from starlight that would contribute to the photon ring or fall in the event horizon per unit time.