# 2022 Online Physics Olympiad: Invitational Contest

# Theoretical Examination

## Sponsors

This competition could not be possible without the help of our sponsors, who are all doing great things in physics, math, and education.

## Instructions for Theoretical Exam

The theoretical examination consists of 5 long answer questions and 110 points over 2 full days from July 30, 0:01 am GMT.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

## Problems

- **T1: Maxwell's Demon**
- **T2: Euler's Disk**
- **T3: Rocket**
- **T4: Magical Box**
- **T5: Quantum Computing**

## List of Constants

| | |
|---|---|
| Proton mass | $m_p = 1.67 \cdot 10^{-27}$ kg |
| Neutron mass | $m_n = 1.67 \cdot 10^{-27}$ kg |
| Electron mass | $m_e = 9.11 \cdot 10^{-31}$ kg |
| Avogadro's constant | $N_0 = 6.02 \cdot 10^{23}\ \mathrm{mol}^{-1}$ |
| Universal gas constant | $R = 8.31\ \mathrm{J/(mol \cdot K)}$ |
| Boltzmann's constant | $k_B = 1.38 \cdot 10^{-23}$ J/K |
| Electron charge magnitude | $e = 1.60 \cdot 10^{-19}$ |
| 1 electron volt | $1\ \mathrm{eV} = 1.60 \cdot 10^{-19}$ J |
| Speed of light | $c = 3.00 \cdot 10^{8}$ m/s |
| Universal Gravitational constant | $G = 6.67 \cdot 10^{-11}\ (\mathrm{N \cdot m^2})/\mathrm{kg}^2$ |
| Acceleration due to gravity | $g = 9.81\ \mathrm{m/s^2}$ |
| 1 unified atomic mass unit | $1\ \mathrm{u} = 1.66 \cdot 10^{-27}\ \mathrm{kg} = 931\ \mathrm{MeV/c^2}$ |
| Planck's constant | $h = 6.63 \cdot 10^{-34}\ \mathrm{J \cdot s} = 4.41 \cdot 10^{-15}\ \mathrm{eV \cdot s}$ |
| Permittivity of free space | $\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}$ |
| Coulomb's law constant | $k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}\ (\mathrm{N \cdot m^2})/\mathrm{C}^2$ |
| Permeability of free space | $\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T \cdot m/A}$ |
| Stefan-Boltzmann constant | $\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2/K^4}$ |

# T1: Maxwell's Demon

Zed has a container divided by a wall into two chambers of equal volume $V$. The left chamber has $N_1$ molecules and the right chamber has $N_2$ molecules of some monatomic ideal gas ($N_1 < N_2$). Each gas molecule has mass $m$ and can be treated as a point particle. The entire system is isolated and is at temperature $T$.

(a) **(5 pts.)** Let's say that he makes a hole in the wall. Then there will be a net flow of molecules from the right chamber to the left chamber. At equilibrium, let's say each chamber has $N = (N_1 + N_2)/2$ molecules. By how much has the entropy increased?

Zed now wants to revert the container back to its original state with $N_1$ and $N_2$ molecules in each chamber. He plans to achieve this by covering the hole with a door with area $A$ that only opens towards the second chamber.

(b) **(5 pts.)** He thinks that any molecule in the left chamber incident on the door will enter the right chamber, and no molecules in the right chamber will enter the left one. Under such a model, what is the initial rate of change in entropy of the system?

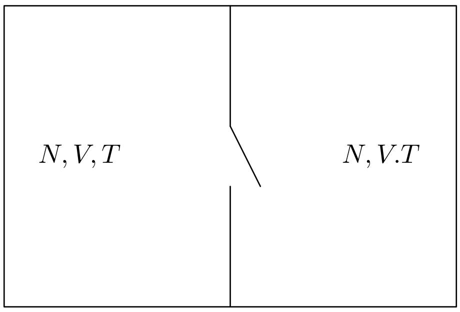



Figure 1: Parts (c) and (d)

Under the assumptions made by part (b), Zed's device violates the second law of thermodynamics. We'll now investigate why this actually does not happen for a particular kind of door. This door, of mass $M$, has a hinge that exerts a restoring torque $\tau = K\theta$ when the door is open at an angle $\theta$, where $\theta$ is not necessarily small (Figure 1).

(c) **(5 pts.)** Explain in one or two sentences why this door behaves effectively like a hole in the wall with area $A'$, and hence the second law of thermodynamics is not violated.

(d) **(10 pts.)** Estimate $A'$ in terms of the variables given and fundamental constants. You may make appropriate simplifying assumptions.

## T2: Euler's Disk

A thin, uniform disk of mass $m$ and radius $a$ is initially set at an angle $\alpha_0$ to the horizontal, on a frictionless surface. It is given an initial angular velocity $\Omega_0$ with respect to a vertical axis passing through its center.

(a) **(4 pts.)** Determine $\Omega_0$ for the steady state case, where $\dot{\alpha} = \ddot{\alpha} = \dot{\Omega} = 0$.

(b) **(2 pts)** Write an expression for the total energy of the disk.

The disk is then moved onto a special surface with small bumps of height $h$ spread over it − each bump is separated by $\delta$. As the disk climbs over a bump and falls back down, its impact is absorbed by the surface, causing a net energy loss in the system. The disk is set in motion with the same initial conditions as before but with $\alpha_0 << 1$

(c) **(6 pts.)** Assuming that this is the only source of energy loss, write a differential equation for $\dot{\alpha}$ in first order to $\alpha$.

(d) **(4 pts.)** Hence, write an approximate expression for $\Omega$ as a function of time.

(e) **(2 pts.)** Using this model, determine the time it takes for the frequency of the sound the disk makes against the surface to reach the maximum audible frequency $f_0$.

## T3: Rocket

OPhO organizers have a "propulsionless" rocket, which for simplicity can be assumed to be a 2-dimensional rectangular box of mass $2M$ and horizontal length $L$. Assume that the horizontal sides of the box are massless while the vertical sides of the box each have mass $M$. The rocket is initially at rest. We will now explore the mechanism for how this rocket move. Suppose we have $N$ particles of mass $m/N$ each on the left and right sides of the box. At time $t = 0$, we launch the $N$ particles on the left side of the box together to the right with velocity $\frac{v}{N}$. In addition, in intervals of time $\frac{L}{v}$, starting at $t = 0$, we launch a particle from the right side of the box to the left side with velocity $v$. Once a particle reaches the opposite side of the box, it is stopped. The particular mechanism to shoot and catch the particles can be ignored here. Assume that this mechanism can conserve energy. After time $t = \frac{NL}{v}$, there will be $N$ particles on each side of the box, which is identical to the initial state.

Neglect relativistic effects in part (a) only.

(a) **(1 pt.)** According to classical (Newtonian) mechanics, what happens to the rocket? Does it move?

(b) **(5 pts.)** If $v \ll c$, how far does the rocket move? Answer in lowest nonzero order in $v/c$.

(c) **(10 pts.)** How far does the center of mass of the rocket system move? Once again, answer in lowest nonzero order in $v/c$. Justify your answer.

(d) **(6 pts.)** Explain why this process cannot continue indefinitely. If it could continue forever, we would able to move the rocket indefinitely with no propulsion.

(e) **(5 pts.)** Give an estimate for how long this process can continue. How far does the rocket move in this time?

## T4: Magical Box

A cubical box of mass $M$ and side length $L$ sits on a horizontal, frictionless plane. The box is filled with an ideal gas of particle mass $m$, particle volume density $n$, and initial temperature $T_0$. One of the vertical walls inside the cube is made of a highly conductive material, kept at a constant temperature $T_b \gg T_0$. The wall is so conductive that the temperature of gas instantaneously changes to $T_b$ after rebounding. All other walls are made of ideal insulators.

(a) **(1 pt.)** State, with a reasoning, the direction in which the box will start moving.

(b) **(7 pts.)** Approximate the initial acceleration $a_0$ of the box. For this question, make sure your equation is valid for $T_b = T_0$ as well.

(c) **(3 pts.)** The acceleration of the box then decreases from $a_0$ to $a_f$ for a short time until $t = \tau_0$. Determine $a_f$.

(d) **(3 pts.)** If $\tau_1$ is the time it takes for acceleration to level off for an identical box with the conductive wall at temperature $\frac{T_b}{3}$, calculate $\frac{\tau_1}{\tau_0}$.

# T5: Quantum Computing

In this problem, you will learn the fundamentals of quantum computers, as well as the physics on how they can be constructed! We have tried to provide as much background information as necessary, but if you believe some part is missing or unclear, please fill out the clarifications form.

## Introduction

Physicists use *braket* notation to describe vectors in quantum systems. When using a vector $\vec{v}$ to describe a quantum state, the *ket*, written as $|v\rangle$ can be used. Both notations below are equivalent:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The *bra*, on the other hand, is the conjugate transpose of the ket $\langle v| = (|v\rangle)^\dagger$. Given two vectors $|v\rangle$ and $|w\rangle$, the *braket* $\langle v|w\rangle = |v\rangle \cdot |w\rangle$ is the inner product of both vectors. This notation will be used throughout this problem.

In any digital device, information is communicated via 0s and 1s, or binary code. The simplest units of this information are called bits. Similar to a bit, the *qubit* can be represented as a linear combination of two orthogonal states: quantum-0 and quantum-1, which are typically $|0\rangle$ and $|1\rangle$ . Here,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Typically, we write a single qubit state as

$$|\Psi\rangle = a|0\rangle + b|1\rangle,$$

where $a, b \in \mathbb{C}$, and $\langle \Psi \rangle \Psi = 1$.

(a) **(1 pt.)** A qubit is prepared in the state $a|0\rangle + b|1\rangle$ .

(i) What is the probability of measuring the qubit in the state $|0\rangle$?

(ii) What is the probability of measuring the qubit in the state $|-\rangle = |0\rangle - |1\rangle$?

*Hint:* If you are still confused about measurement (it's tricky!), check out this qiskit article. You can ignore all the parts with code, we'll save those for the computer science students writing OCSO.

A quantum gate performs an unitary operator on a quantum state. Applying an operator (sometimes known as a gate) to a qubit state can be represented in the diagram below.

$$|\Psi\rangle \quad —\boxed{\hat{U}}— \quad \hat{U}|\Psi\rangle$$

where $\hat{U}$ is a local unitary since it only acts on a single qubit. There are five important gates:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here, $I, X, Y, Z$ form the four **Pauli matrices** and $H$ is known as the **Hadamard** gate, which we will use later on when we talk about entanglement.

For example, if $|\Psi\rangle = 0.6|0\rangle + 0.8|1\rangle$ and apply the gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we end up with

$$X|\Psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = 0.8|0\rangle + 0.6|1\rangle.$$

(b) **(1 pt.)** A qubit is prepared in the state $|\Psi\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring the qubit $\hat{U}|\Psi\rangle$ in the state $|0\rangle$? Express your answer in terms of $a, b$, and properties of the unitary $\hat{U}$.

The heart of quantum information lies in what we can do with more than a single qubit. If one qubit has two dimensions ($|0\rangle$ and $|1\rangle$), then a two-qubit system can be represented in four dimensions. For a two qubit system, the state can be written as $a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$, where $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ can be seen as the "basis vectors." If we have two independent qubits, i.e. $|\Psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\Psi_2\rangle = c|0\rangle + d|1\rangle$, we can represent their combined state using the **tensor product**, i.e.

$$\begin{aligned} |\Psi\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle . \end{aligned}$$

Here, we can see that

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Note that not every two qubit state can be written as a tensor product. When this occurs, we say that they are **entangled.** We can immediately determine if a state is entangled by calculating its **concurrence,** defined by

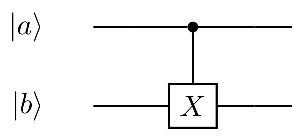
$$C = 2|a_0 a_3 - a_1 a_2|.$$

If $C = 0$, then the two qubits are separate and the system is **separable.** If $C = 1$, the system is maximally entangled, such as

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle .$$

Physically, this means that a measurement of one qubit directly leads to a "collapse" of the other qubit (this is the classic example shown in popular science media). Note that $0 \leq C \leq 1$.

We can change the concurrence using a **control operation.** For example,



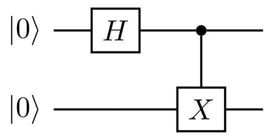

performs the **CNOT** gate. The unitary $X$ is applied to $|b\rangle$ if $|a\rangle = 1$, otherwise nothing is done. That is, we have:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle . \end{aligned}$$

The CNOT gate is an example of a global unitary, since it acts on more than one qubit. Global unitaries for 2 qubit systems can be written as a $4 \times 4$ matrix. For example, we can write

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $|00\rangle, \ldots, |11\rangle$ form the 4 standard basis vectors. We can combine local and global unitaries to create entangled states. For example, consider the following circuit:




The initial state is $|\Psi_{\text{in}}\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. After applying the Hadamard gate $H$, the state becomes

$$|\Psi_{\text{middle}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}.$$

After applying the CNOT gate, the state becomes:

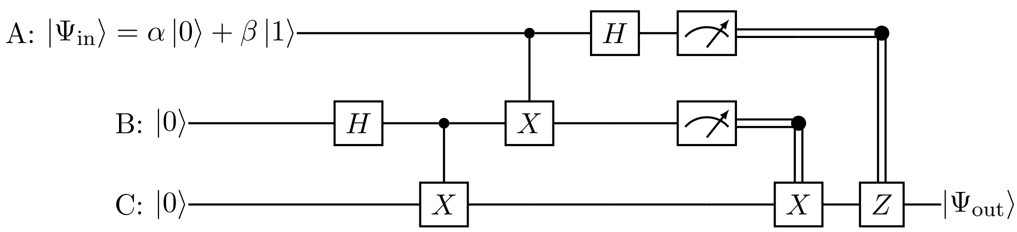
$$|\Psi_{\text{out}}\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

Note that we can avoid matrix multiplication in this last step by seeing what CNOT does on each term of $|\Psi_{\text{middle}}\rangle$. CNOT will not have an effect on $\frac{1}{2}|00\rangle$ since the first qubit is $|0\rangle$. CNOT will have an effect on $\frac{1}{2}|10\rangle$ since the first qubit is a $|1\rangle$, so it'll flip the second qubit to a $|1\rangle$, giving us the map $\frac{1}{2}|10\rangle \mapsto \frac{1}{2}|11\rangle$.

(c) **(1 pt.)** Construct a quantum circuit where the input state is $|00\rangle$ and the output state is $\frac{i}{\sqrt{2}}(|0\rangle - |1\rangle)$ using only $X, Y, Z, H, CNOT$ gates.

## Quantum Teleportation

Quantum teleportation is the transfer of the quantum state of one qubit to another (not the actual physical qubit) using a shared entangled resource and two classical bits of information. It is performed using the following circuit.



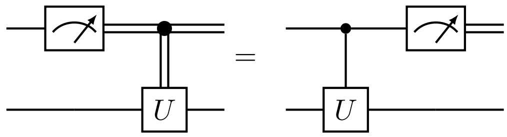

The [measurement gate symbol] gate measures the qubit (returns either a 0 or a 1) and the wider wire represents that information that flows through this wire is a classical bit.

(d) **(1 pt.)** Verify that the above circuit does teleport the qubit from the top branch to the bottom branch by looking at the specific case of $\alpha = \beta = \frac{1}{\sqrt{2}}$

(e) **(3 pts.)** After the first operation is performed on branch $C$, the branch is brought a very far distance from the other two branches. By doing so, it appears we can create faster-than-light communication during the teleportation process, which is impossible! Explain why there is no contradiction. Justify rigorously.

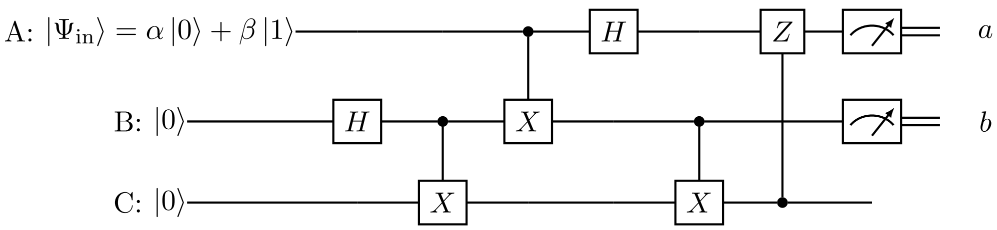
We can analyze this by performing matrix multiplication, but using a circuit-based approach is much cleaner. To do so, we need to use the **Griffiths-Niu Theorem.**

(f) **(2 pts.)** The following circuits, according to the Griffiths-Niu Theorem, are equivalent:

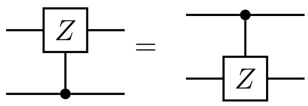



Prove the Griffiths-Niu Theorem.

Using this theorem, we can redraw our circuit as:

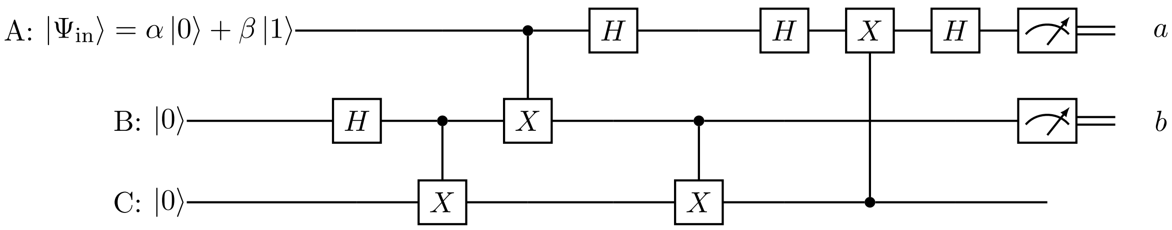



(g) **(1 pts.)** For a control-Z gate, it doesn't matter which branch is the control. In other words,

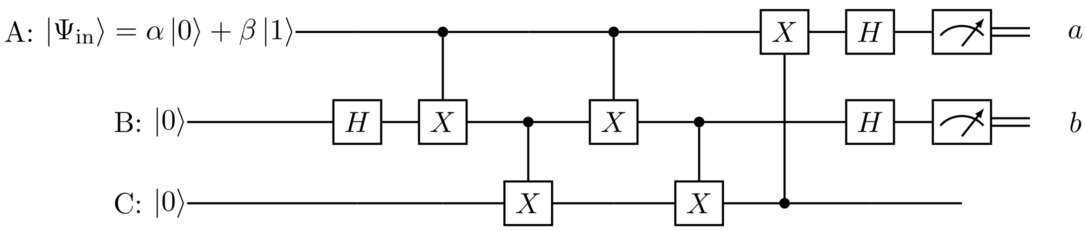



Prove this relationship.

Using the above problem, we can flip the control-Z gate. Then using the identity $Z = HXH$, we can reduce it further:

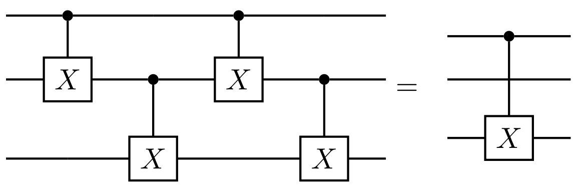



Since $H^2 = I$, we can simplify the top part. Furthermore, we can introduce another CNOT between the first and the second branch.

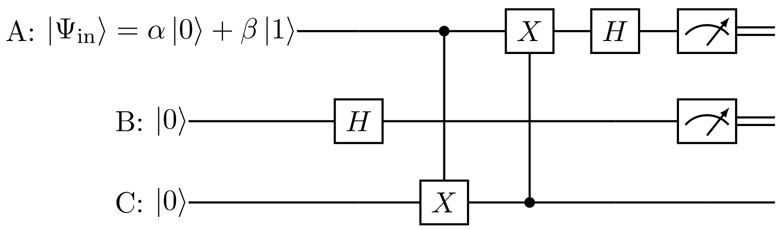



We were allowed to introduce this CNOT gate since $XH|0\rangle = H|0\rangle$ . This actually makes it easier using the following problem:

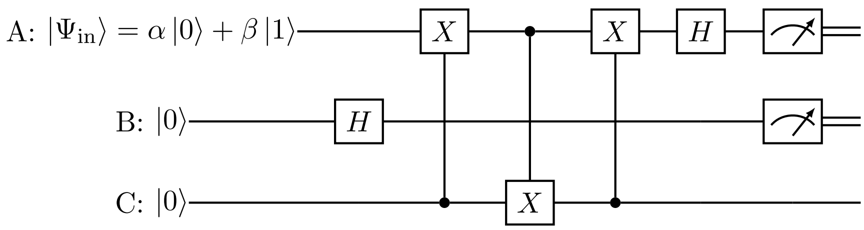
(h) **(2 pts.)** Prove that the below two circuits are equivalent.



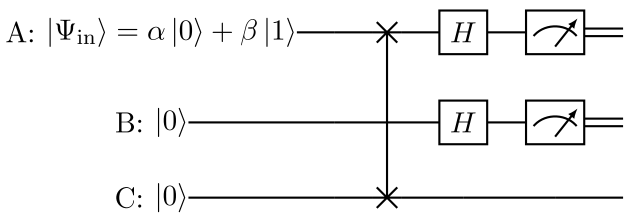

Using this substitution, we end up with:




We can now introduce another CNOT gate, which doesn't do anything since $C$ will always be $|0\rangle$.




Three alternating CNOT gates is equivalent to the SWAP gate, so we can write:




where we clearly see a swapping that occurs between the top and bottom branch!

## Building Quantum Computers

According to theoretical physcist David P. Divencenzo, there are five necessary (but not necessarily sufficient) criteria to build a quantum computer:

- A well-characterized qubit.
- The ability to initialize qubits.
- Long and relevant decoherence times.
- A "universal set" of quantum gates.
- The ability to measure qubits.

In this section, we will focus on how we can create qubits and how we can create a universal set of quantum gates. Consider two energy levels $E_1, E_0$ as the qubit states $|1\rangle, |0\rangle$ respectively. Assume that

$$E_1 = \frac{1}{2}\hbar\omega, \qquad E_0 = -\frac{1}{2}\hbar\omega.$$

Also assume that the qubit state is time varying, in the form of:

$$|\Psi(t)\rangle = A(t)\,|0\rangle + B(t)\,|1\rangle\,.$$

(i) **(14 pts.)** Using the above setup, show how we can implement the quantum gates $X, Y, Z$. *Hint:* The Schrodinger Equation tells us

$$i\hbar\frac{\partial}{\partial t}\,|\Psi(t)\rangle = \hat{\mathcal{H}}\,|\Psi(t)\rangle\,,$$

where $\hat{\mathcal{H}} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$.